Is completeness necessary? Estimation in nonidentified linear models*

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Abstract

Estimators based on spectral regularization converge to the best approximation to the structural parameter even when the completeness condition fails. The convergence holds in the underlying Hilbert space and supremum norms. The best approximation may or may not coincide with the structural parameter of interest, depending on whether it has a component in the null space of the operator or not. We show that identification failures can affect the asymptotic distribution of linear functionals with a chi-squared component. The theory is illustrated for high-dimensional and nonparametric IV regressions.

Keywords: completeness condition, nonidentified models, weak identification, nonparametric IV regression, high-dimensional regressions, spectral regularization.

JEL Classifications: C14, C26

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1 Introduction

Structural nonparametric and high-dimensional models are often ill-posed. Among many examples, we may quote the nonparametric IV regression, various high-dimensional regressions, measurement errors, and random coefficient models. All these examples lead to the ill-posed functional equation

$$K\varphi = r,$$

where φ is a structural parameter of interest, r is a function, and K is a linear operator. The classical numerical literature on ill-posed inverse problems, see Engl, Hanke, and Neubauer (1996), studies *deterministic* problems, where the operator K is usually known and r is measured with a deterministic numerical error. In econometric applications, both K and r are estimated from the data, and we are faced with the *statistical* ill-posed inverse problems.

Identification is an integral part of econometric analysis, going back to Koopmans (1949), Koopmans and Reiersol (1950), and Rothenberg (1971) in the parametric case. In nonparametric and high-dimensional ill-posed inverse problems, K and r are directly identified from the data-generating process, while the structural parameter φ is identified if the equation $K\varphi = r$ has a unique solution. The uniqueness is equivalent to assuming that K is a one-to-one operator, or in other words, that $K\phi = 0 \implies \phi = 0$ for all ϕ in the domain of K. It is worth stressing that the operator K is usually unknown in econometric applications and the estimated operator $\hat{K}$ has a finite rank and is not one-to-one for every finite sample size.

The maximum likelihood estimator when there is a lack of identification leads to a flat likelihood in some regions of a parameter space and then to ambiguity in the choice of maximum. It is then natural to characterize the limit of an estimator for such a potentially nonidentified model. In the nonidentified ill-posed inverse models, the identified set is a linear manifold $\phi + \mathcal{N}(K)$, where ϕ is any solution to $K\phi = r$, and $\mathcal{N}(K)$ is the null space of K. Note that the identified set is, in general, unbounded and is not informative on the structural parameter φ without additional constraints.

As K or $\hat{K}$ may fail to be one-to-one and typically have a discontinuous generalized inverse, some regularization is needed to estimate consistently the structural parameter. In this paper, we focus on spectral regularization methods consisting of modifying the spectrum of the operator $\hat{K}$. Tikhonov regularization, also known as functional ridge regression, is one prominent example; see Tikhonov (1963). Other important instances of spectral regularizations include the iterated Tikhonov; the spectral cut-off, which is also related to functional principal component analysis, see Yao, Müller, and Wang (2005); and the Landweber iteration, which is also related to functional gradient descent.[1] We show that estimators based on spectral regularization are uniformly consistent for the best approximation to the structural parameter in the orthogonal complement to the null space of the operator and study the distributional consequences of identification failures.

In some cases, the best approximation may coincide with the structural parameter or at least may reasonably approximate it, even when the operator K has a non-trivial null space. This provides an attractive interpretation for the nonparametric IV regression under identification failures, similar in a way to the best approximation property of least-squares under misspecifications; see Angrist and Pischke (2008), Chapter 3.[2] Lastly, the best approximation can also be used to construct set estimators in the partial identification approach.

Contribution and related literature. Our paper contributes to several strands of literature. First, there exists a large literature on the regularization of statistical inverse problems; see Newey and Powell (2003), Blundell, Chen, and Kristensen (2007), Chen and Pouzo (2012), Chen and Christensen (2018), Carrasco, Florens, and Renault (2007, 2014), Darolles, Fan, Florens, and Renault (2011), Florens, Johannes, and Van Bellegem (2011), Gagliardini and Scaillet (2012), Hall and Horowitz (2005), and Babii (2020, 2022), among others. Canay, Santos, and Shaikh (2013) show that the identifying completeness condition is not testable in these models; see also D'Haultfoeuille (2011), Hu and Shiu (2018), and Andrews (2017) for various primitive conditions. Our contribution to this literature is to derive the Hilbert space and L_∞ convergence rates under easy-to-verify conditions for estimates based on *generic spectral regularization schemes* including Tikhonov, iterated Tikhonov, Landweber iteration, and spectral cut-off. These results do not require the completeness condition and are illustrated for the nonparametric IV and the functional linear IV regressions.[3]

Second, there is an important strand of literature on the identification of linear functionals for the nonparametric IV regression. For example, Severini and Tripathi (2006, 2012) show that linear functionals can be identified under conditions that are weaker than the completeness condition and derive the corresponding efficiency

[1] Functional gradient descent is also a building block of various boosting estimators, see Friedman (2001).

[2] In contrast, the parametric 2SLS estimator does have the best approximation interpretation; see Escanciano and Li (2021) for an example of the IV estimator that has such an interpretation.

[3] These results provide a substantial generalization of Florens, Johannes, and Van Bellegem (2011) who cover *only* the L_2 rates for the Tikhonov regularization under some high-level conditions.

bounds. Santos (2011) establishes the asymptotic normality of linear functionals without requiring the completeness condition. Escanciano and Li (2021) obtain necessary and sufficient conditions for the identification and $\sqrt{n}$-estimability of the best linear approximation to the nonparametric IV functional. The condition of Escanciano and Li (2021) is also weaker than the completeness condition. Our contribution to this literature is to show that the asymptotic distribution of generic linear functionals can have a chi-squared component when the completeness condition fails and the functional is estimable at a slower than $\sqrt{n}$ rate.

Third, our paper connects to the partial identification literature in nonparametric IV regression; see Santos (2012), Freyberger (2017), and Chernozhukov, Newey, and Santos (2023). Our results on the consistency to the best approximation in the L_∞ norm, in conjunction with Freyberger (2017), can be used for partial identification.

The paper is organized as follows. Section 2 briefly discusses identification in linear ill-posed inverse models. In Section 3, we obtain the non-asymptotic risk bounds in the L_∞ and Hilbert space norms for a class of Tikhonov-regularized estimators. These results are generalized to other spectral regularization schemes (including iterated Tikhonov, spectral cut-off, and Landweber iterations) in the Appendix Section A.1. Building on these results, we illustrate in Section 4 the transition between the Gaussian and the weighted chi-squared asymptotics when the completeness condition fails. We report on a Monte Carlo study in Section 5, which provides further insights about the validity of our asymptotic results in finite sample settings typically encountered in empirical applications. Section 6 concludes. All proofs are collected in the Appendix; see Section A.2. Appendix Section A.3 describes how our results can be used for partial identification. In Appendix Section A.4, we show that in the extreme case of identification failures, the Tikhonov-regularized estimator is driven by degenerate U-statistics in large samples. Lastly, we review several relevant results from the theory of generalized inverse operators and the theory of Hilbert space-valued U-statistics in the Online Appendix Sections B.1 and B.2.

2 Identification

Consider the functional linear equation

$$K\varphi = r,$$

where $K : \mathcal{E} \to \mathcal{H}$ is a compact linear operator, defined on some Hilbert spaces $\mathcal{E}$ and $\mathcal{H}$, and $\varphi \in \mathcal{E}$ is a structural parameter of interest. The structural parameter φ is point identified if the operator K is one-to-one, or in other words if the null space of K, denoted $\mathcal{N}(K) = \{\phi \in \mathcal{E} : \ K\phi = 0\}$, reduces to $\{0\}$. Equivalently, the point identification of φ requires that

$$K\phi = 0 \implies \phi = 0, \qquad \forall \phi \in \mathcal{E}.$$

If the completeness condition fails, then the identified set is a closed linear manifold, denoted $I_0 = \varphi + \mathcal{N}(K)$, where $\mathcal{N}(K)$ is the null space of K. Since $\mathcal{N}(K)$ is a closed linear subspace of $\mathcal{E}$, decompose $\varphi = \varphi_1 + \varphi_0$, where φ_1 is the unique projection of φ on $\mathcal{N}(K)^\perp$ and φ_0 is the orthogonal projection of φ on $\mathcal{N}(K)$. Since $\mathcal{N}(K)^\perp = \overline{\mathcal{R}(K^*)}$, as shown in Luenberger (1997), p.157, the best approximation φ_1 equals to the structural parameter φ whenever the structural parameter φ belongs to the closure of the range of the adjoint operator to K, $\overline{\mathcal{R}(K^*)}$. This condition also has an appealing regularity interpretation known as the source condition. To see this, note that $\mathcal{R}(K^*) = \mathcal{R}(K^*K)^{1/2}$; see Engl, Hanke, and Neubauer (1996), Proposition 2.18. Therefore, if the ill-posed inverse problem has a sufficiently high regularity, so that $\varphi \in \mathcal{R}(K^*K)^{\beta/2}$ with $\beta \in [1, \infty)$, then $\varphi_1 = \varphi$, and the structural function φ is point identified even though the completeness condition fails. Hence, the completeness condition is not needed if additional regularity assumptions are imposed on the structural function. The following two examples further illustrate this point.[4]

Example 2.1 (Nonparametric IV regression)**.** *Consider*

$$Y = \varphi(Z) + U, \qquad \mathbb{E}[U|W] = 0,$$

where $(Y, Z, W) \in \mathbf{R} \times \mathbf{R}^p \times \mathbf{R}^q$*. The exclusion restriction leads to*

$$r(w) \triangleq \mathbb{E}[Y|W = w] = \mathbb{E}[\varphi(Z)|W = w] \triangleq (K\varphi)(w),$$

where $K : L_2(Z) \to L_2(W)$ *is a conditional expectation operator.*[5] *If* K *is compact, then by the spectral theorem, there exists* $(\sigma_j, e_j, h_j)_{j\geq 1}$*, where* $\sigma_j \to 0$ *is a sequence of singular values,* $(e_j)_{j\geq 1}$ *is a complete orthonormal system of* $\mathcal{N}(K)^\perp = \overline{\mathcal{R}(K^*)}$*, and* $(h_j)_{j\geq 1}$ *is the complete orthonormal system of* $\mathcal{N}^\perp(K^*) = \overline{\mathcal{R}(K)}$*; see Kress (2014), Theorem 15.16.* φ_1 *coincides with* φ*, whenever it can be represented in terms of* $(e_j)_{j\geq 1}$*.*

Example 2.2 (Nonparametric IV with discrete instrument)**.** *Following the previous example, suppose that the instrumental variable is discrete,* $W \in \{w_k : \ k \geq 1\}$*. Put* $f_k(z) = f_{Z|W=w_k}(z)$ *for every* $k \geq 1$*. Then*

$$\mathcal{N}(K) = \{\phi \in L_2(Z) : \ \langle \phi, f_k \rangle = 0, \ \forall k \geq 1\}$$

[4] See also Florens, Bontemps, and Nour (2023) for examples of ecological inference.

[5] We use the Hilbert space $L_2(X) = \{\phi : \mathbb{E}|\phi(X)|^2 < \infty\}$.

and if $\varphi \in \operatorname{span}\{f_k : k \geq 1\}$, then $\varphi \in \mathcal{N}(K)^\perp$. φ_1 coincides with φ whenever it can be represented in terms of $(f_k)_{k\geq 1}$.

3 Nonasymptotic Risk Bounds

In this section, we obtain risk bounds for the Tikhonov estimator solving

$$\min_{\phi} \left\| \hat{K}\phi - \hat{r} \right\|^2 + \alpha_n \|\phi\|^2,$$

where $\alpha_n > 0$ is a regularization parameter. It is straightforward to verify that the solution to this problem is

$$\hat{\varphi} = (\alpha_n I + \hat{K}^* \hat{K})^{-1} \hat{K}^* \hat{r}.$$

Note that the estimator $\hat{\varphi}$ is well-defined even if K or $\hat{K}$ are not one-to-one.

3.1 Hilbert Space Risk

First, we describe the relevant class of structural functions and operators:

Assumption 3.1. *The structural parameter $\varphi = \varphi_1 + \varphi_0$ and the operator K are in*

$$\mathcal{F} = \mathcal{F}(\beta, C) = \left\{ (\varphi, K) : \ \varphi_1 = (K^*K)^{\beta/2}\psi, \ \|\psi\|^2 \vee \|\varphi_0\| \vee \|K\| \leq C \right\}$$

for some $\beta, C > 0$, where $\|K\| = \sup_{\|\phi\|\leq 1} \|K\phi\|$ and $a \vee b = \max\{a, b\}$.

To illustrate this assumption, let $(\sigma_j, e_j, h_j)_{j\geq 1}$ be the spectral decomposition of $K : \mathcal{E} \to \mathcal{H}$; see Kress (2014), Theorem 15.16. Then $\varphi_1 = \sum_{j\geq 1} \langle \varphi_1, e_j \rangle e_j$ and by the Parseval's identity, since $\psi = (K^*K)^{-\beta/2}\varphi_1$, we have

$$\|\psi\|^2 = \sum_{j\geq 1} \frac{|\langle \varphi_1, e_j \rangle|^2}{\sigma_j^{2\beta}}.$$

Therefore, Assumption 3.1 restricts the relative rates of decline of singular values $(\sigma_j)_{j\geq 1}$, describing the ill-posedness, and Fourier coefficients $\langle \varphi_1, e_j \rangle_{j\geq 1}$, describing the regularity of φ_1.

We estimate (r, K) with $(\hat{r}, \hat{K})$ and make the following assumption:

Assumption 3.2. *(i) $\mathbb{E}\|\hat{r} - \hat{K}\varphi_1\|^2 \leq C_1\delta_n$; and (ii) $\mathbb{E}\|\hat{K} - K\|^2 \leq C_2\rho_{1n}$, where $C_1, C_2 < \infty$ do not depend on (φ, K) and $\delta_n, \rho_{1n} \to 0$ as $n \to \infty$.*

Assumption 3.2 restricts the convergence rate of residuals and the estimated operator. Note that Florens, Johannes, and Van Bellegem (2011), Theorem 2.2 imposes a stronger high-level condition

$$\mathbb{E}\|\hat{K}-K\|^4 \leq C\left(\mathbb{E}\|\hat{K}-K\|^2\right)^2$$

and some assumptions on the tuning parameters that have not been verified for specific models.

The following result holds for the Hilbert space norm:

Theorem 3.1. *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then for all* $\beta \in (0,2]$

$$\sup_{(\varphi,K)\in\mathcal{F}} \mathbb{E}\left\|\hat{\varphi}-\varphi_1\right\|^2 = O\left(\frac{\delta_n + \rho_{1n}\alpha_n^{\beta\wedge 1}}{\alpha_n} + \alpha_n^{\beta}\right),$$

where $a \wedge b = \min\{a,b\}$.

Theorem 3.1 shows that the convergence rate in the Hilbert space norm is driven by the rates of:

- residuals, $O\left(\delta_n/\alpha_n\right)$;
- estimated operator, $O\left(\rho_{1n}\alpha_n^{\beta\wedge 1}/\alpha_n\right)$;
- the regularization bias, $O\left(\alpha_n^{\beta}\right)$.

It is worth mentioning a stronger version of Assumption 3.1 is commonly assumed: $\varphi \in \mathcal{R}(K^*K)^{\beta/2}$ with $\beta \geq 1$; see Carrasco, Florens, and Renault (2007). In this case, we actually have $\varphi_0 = 0$ and the Tikhonov regularized estimator can estimate consistently the structural parameter φ, despite the fact that the completeness condition fails. More generally, we distinguish the following possibilities:

- identified case: $\varphi_0 = 0$ and $\rho_{1n}\alpha_n^{\beta\wedge 1} \lesssim \delta_n$, where the convergence rate to φ is driven by residuals;
- weakly identified case: $\varphi_0 = 0$ and $\delta_n \lesssim \rho_{1n}\alpha_n^{\beta\wedge 1}$, where the convergence rate to φ is driven by the estimated operator;
- nonidentified case: $\varphi_0 \neq 0$ and $\rho_{1n}\alpha_n^{\beta\wedge 1} \lesssim \delta_n$, where the convergence rate to φ_1 is driven by residuals;
- strongly nonidentified models: $\varphi_0 \neq 0$ and $\delta_n \lesssim \rho_{1n}\alpha_n^{\beta\wedge 1}$, where the convergence rate to φ_1 is driven by the estimated operator.

In the identified case, the optimal choice of regularization parameter is $\alpha_n \sim \delta_n^{1/(\beta+1)}$ leading to the rate of order $O\left(\delta_n^{\beta/(\beta+1)}\right)$ with $\beta \in (0,2]$. For $\beta > 2$, we show in the Appendix Section A.1 that the optimal rate can be achieved with additional iterations or some alternative regularization schemes.

Lastly, it is worth mentioning that in the special case of L_2 spaces, Theorem 3.1 provides a sharper result than Florens, Johannes, and Van Bellegem (2011), Theorem 2.2, where the rate is always driven by ρ_{1n}.

3.2 L_∞ Risk

Suppose now that the space of continuous functions on a compact set $D \subset \mathbf{R}^p$, denoted $(C(D), \|.\|_\infty)$, is embedded into the space $\mathcal{E}$, where $\|.\|_\infty$ is the uniform norm. Suppose also that $\mathcal{R}(K^*) \subset C(D)$ and that $\varphi_1 \in C(D)$. Let $\|K^*\|_{2,\infty} = \sup_{\|\phi\|\leq 1} \|K^*\phi\|_\infty$ be the mixed operator norm of K^*. The following assumption describes how well the operator K^* is estimated by $\hat{K}^*$ in the $\|.\|_{2,\infty}$ norm.

Assumption 3.3. *Suppose that $\|K^*\|_{2,\infty} \leq C_3$ and that $\mathbb{E}\|\hat{K}^* - K^*\|^2_{2,\infty} \leq C_3\rho_{2n}$, where $C_3 < \infty$ does not depend on (φ, K) and $\rho_{2n} \to 0$.*

The following result holds:

Theorem 3.2. *Suppose that Assumptions 3.1, 3.2, and 3.3 are satisfied with $\varphi_1 = (K^*K)^{\beta/2}K^*\psi$. Then for all $\beta \in (0,2]$*

$$\sup_{(\varphi,K)\in\mathcal{F}} \mathbb{E}\,\|\hat{\varphi} - \varphi_1\|_\infty = O\left(\frac{\delta_n^{1/2} + \rho_{1n}^{1/2}\alpha_n^{\frac{\beta+1}{2}\wedge 1} + \rho_{2n}^{1/2}\alpha_n^{1/2}}{\alpha_n} + \alpha_n^{\beta/2}\right).$$

Theorem 3.2 describes the uniform convergence rates for generic ill-posed inverse problems. In constrast to Theorem 3.1, the L_∞ convergence rate is influenced by the estimation error in K^*, measured in the $\|.\|_{2,\infty}$ norm; see also Appendix Section A.1, for the L_∞ convergence rate of more general regularized estimators.

3.3 Applications

3.3.1 High-dimensional Regressions

Consider

$$Y = \langle Z, \varphi\rangle + U, \qquad \mathbb{E}[UW] = 0,$$

where $(Y, Z, W) \in \mathbf{R} \times \mathcal{E} \times \mathcal{H}$, see Florens and Van Bellegem (2015).[6] The exclusion restriction leads to the functional linear equation

$$r \triangleq \mathbb{E}[YW] = \mathbb{E}[\langle Z, \varphi \rangle W] \triangleq K\varphi,$$

where $K : \mathcal{E} \to \mathcal{H}$ is a covariance operator. An econometrician observes an i.i.d. sample $(Y_i, Z_i, W_i)_{i=1}^n$.[7] Then

$$r = \mathbb{E}[YW], \qquad K\phi = \mathbb{E}[W\langle \phi, Z\rangle], \qquad K^*\psi = \mathbb{E}[Z\langle \psi, W\rangle]$$

are estimated with

$$\hat{r} = \frac{1}{n}\sum_{i=1}^n Y_i W_i, \qquad \hat{K}\phi = \frac{1}{n}\sum_{i=1}^n W_i\langle Z_i, \phi\rangle, \qquad \hat{K}^*\psi = \frac{1}{n}\sum_{i=1}^n Z_i\langle W_i, \psi\rangle.$$

It is easy to verify that

$$\mathbb{E}\left\|\hat{r} - \hat{K}\varphi_1\right\|^2 = \frac{\mathbb{E}\|(Y - \langle Z, \varphi_1\rangle)W\|^2}{n} \qquad \text{and} \qquad \mathbb{E}\left\|\hat{K} - K\right\|^2 \leq \frac{\mathbb{E}\|ZW\|^2}{n}.$$

Let $\mathcal{F}(\beta, C)$ be the class of models as in the Assumption 3.1 and suppose also that $\mathbb{E}\|UW\|^2 \vee \mathbb{E}\|ZW\|^2 \leq C$ for all models in this class. Then $\delta_n = \rho_{1n} = n^{-1}$ and Theorem 3.1 shows that the Hilbert space risk is of order

$$\sup_{(\varphi, K)\in\mathcal{F}} \mathbb{E}\left\|\hat{\varphi} - \varphi_1\right\|^2 = O\left(\frac{1}{\alpha_n n} + \alpha_n^\beta\right).$$

The high-dimensional regression is either identified ($\varphi_0 = 0$) or nonidentified ($\varphi_0 \neq 0$). Then conditions $\alpha_n \to 0$ and $\alpha_n n \to \infty$ as $n \to \infty$ are sufficient to guarantee consistency in the Hilbert space norm. In particular, the optimal choice $\alpha_n \sim n^{-1/(\beta+1)}$ leads to the convergence rate of order $O(n^{-\beta/(\beta+1)})$. This rate is nearly optimal in the special case when $W = Z$; see Babii, Carrasco, and Tsafack (2024), Theorem 2.2.

[6]For instance, when $W = Z$ and $\mathcal{E} = L_2$ with counting measure, we obtain a high-dimensional regression model, suitable for non-sparse data

$$Y = \sum_{j\geq 1}\varphi_j Z_j + U, \qquad \mathbb{E}[UZ_j] = 0, \qquad \forall j \geq 1.$$

When $\mathcal{E} = L_2$ with Lebesgue measure, this model is also sometimes called the functional regression; see Florens and Van Bellegem (2015) and Babii (2022).

[7]The i.i.d. assumption can be relaxed to the covariance stationarity and absolute summability of autocovariances; see Babii (2022).

For the uniform convergence, suppose that $\mathcal{E} = L_2(D)$, i.e. a set of functions on some bounded set $D \subset \mathbf{R}^p$, square-integrable with respect to the Lebesgue measure. To verify the Assumption 3.3, we also assume that models in $\mathcal{F}(\beta, C)$ are such that $\|Z\|_\infty \vee \|W\|_\infty \leq C < \infty$ and that stochastic processes Z and W are in some Hölder ball with smoothness $s > d/2$. Then by the Hoffman-Jørgensen and moment inequalities, see Giné and Nickl (2016), p.129 and p.202,

$$\mathbb{E}\left\|\hat{K}^* - K^*\right\|_{2,\infty}^2 = O\left(\frac{1}{n}\right).$$

Therefore, Assumption 3.3 is satisfied with $\rho_{2n} = n^{-1}$, and Theorem 3.2 shows that

$$\sup_{(\varphi,K)\in\mathcal{F}} \mathbb{E}\left\|\hat{\varphi} - \varphi_1\right\|_\infty = O\left(\frac{1}{\alpha_n n^{1/2}} + \alpha_n^{\beta/2}\right).$$

Then conditions $\alpha_n \to 0$ and $\alpha_n n^{1/2} \to \infty$ as $n \to \infty$ ensure the uniform consistency of $\hat{\varphi}$. In particular, the optimal choice $\alpha_n \sim n^{-1/(\beta+2)}$ leads to the convergence rate of order $O(n^{-\beta/(2\beta+4)})$. It is an open question what is the minimax-optimal convergence rate for the class $\mathcal{F}(\beta, C)$.

3.3.2 Nonparametric IV Regression

Following Example 2.1, rewrite the model as

$$r(w) \triangleq \mathbb{E}[Y|W = w] f_W(w) = \int \varphi(z) f_{ZW}(z, w)\mathrm{d}z \triangleq (K\varphi)(w),$$

where $K : L_2([0,1]^p) \to L_2([0,1]^q)$. We can estimate r and K via kernel smoothing:

$$\begin{aligned}
\hat{r}(w) &= \frac{1}{nh_n^q}\sum_{i=1}^n Y_i K_w\left(h_n^{-1}(W_i - w)\right),\\
(\hat{K}\phi)(w) &= \int \phi(z)\hat{f}_{ZW}(z, w)\mathrm{d}z,\\
\hat{f}_{ZW}(z, w) &= \frac{1}{nh_n^{p+q}}\sum_{i=1}^n K_z\left(h_n^{-1}(Z_i - z)\right) K_w\left(h_n^{-1}(W_i - w)\right),
\end{aligned}$$

where K_w and K_z are symmetric kernel functions and $h_n \to 0$ is a bandwidth parameter. Under mild assumptions, by Proposition A.2.1, $\delta_n = \frac{1}{nh_n^q} + h_n^{2s}$ and

$\rho_{1n} = \frac{1}{nh_n^{p+q}} + h_n^{2s}$, where s is the Hölder smoothness of f_{ZW}. Therefore, Theorem 3.1 shows that mean-integrated squared error has the following rate

$$\sup_{(\varphi,K)\in\mathcal{F}} \mathbb{E}\left\|\hat{\varphi} - \varphi_1\right\|^2 = O\left(\frac{1}{\alpha_n}\left(\frac{1}{nh_n^q} + h_n^{2s}\right) + \frac{1}{nh_n^{p+q}}\alpha_n^{(\beta-1)\wedge 0} + \alpha_n^{\beta}\right),$$

where the class $\mathcal{F}(\beta, C)$ includes additional moment restrictions; see Babii (2020). In the nonparametric IV model, all four identification cases are possible, depending on the value of the regularity parameter β. For consistency of $\hat{\varphi}$ to φ_1, we need $\alpha_n nh_n^q \to \infty$, $\alpha_n^{(1-\beta)\wedge 0} nh_n^{p+q} \to \infty$, and $h_n^{2s}/\alpha_n \to 0$ as $n \to \infty$, $\alpha_n \to 0$, and $h_n \to 0$.

We also know that $\rho_{2n}^{1/2} = \sqrt{\frac{\log h_n^{-1}}{nh_n^{p+q}}} + h_n^s$, see e.g., Babii (2020), Proposition A.3.1. Therefore, Theorem 3.2 shows that

$$\sup_{(\varphi,K)\in\mathcal{F}} \mathbb{E}\|\hat{\varphi} - \varphi_1\|_\infty = O\left(\frac{1}{\alpha_n}\left(\frac{1}{\sqrt{nh_n^q}} + h_n^s\right) + \frac{1}{\alpha_n^{1/2}}\sqrt{\frac{\log h_n^{-1}}{nh_n^{p+q}}} + \alpha_n^{\beta/2}\right).$$

For the uniform consistency of $\hat{\varphi}$ to the best approximation φ_1, we need $\alpha_n^2 nh_n^q \to \infty$, $\alpha_n nh_n^{p+q} \to \infty$, and $h_n^s/\alpha_n \to 0$ as $n \to \infty$, $\alpha_n \to 0$, and $h_n \to 0$.

It remains unclear whether these convergence rates for the NPIV are minimax-optimal for the class $\mathcal{F}(\beta, C)$. Note that this class does not require that the function φ_1 is differentiable or has certain Hölder smoothness; see Chen and Christensen (2018) who prove the minimax optimal convergence rates for the NPIV model on the Hölder ball.

4 Linear Functionals

In some economic applications, the object of interest is a linear functional of the structural function φ. In this section, we show that the asymptotic distribution of such functionals may be non-standard under identification failures. For simplicity of presentation, we focus on the functional IV regression of Florens and Van Bellegem (2015); see also Section A.4 for additional results on extreme identification failures.

By the Riesz representation theorem, any continuous linear functional on a Hilbert space $\mathcal{E}$ can be represented as an inner product with some $\mu \in \mathcal{E}$. Therefore, without loss of generality, we can focus on the asymptotic distribution of $\langle \hat{\varphi}, \mu \rangle$. Consider the unique orthogonal decomposition $\mu = \mu_0 + \mu_1$, where μ_0 is the orthogonal projection on $\mathcal{N}(K)$ and μ_1 is the orthogonal projection on $\mathcal{N}(K)^\perp$. Decompose also $W = W^0 + W^1$, where W^0 is the orthogonal projection of W on $\mathcal{N}(K^*)$ and W^1 is the orthogonal projection of W on $\mathcal{N}(K^*)^\perp$.

The following two assumptions impose some restrictions on the distribution of the data and tuning parameters. Note that the restrictions on the distribution of the data are relatively mild and plausible in economic applications.

Assumption 4.1. *(i) the data* $(Y_i, Z_i, W_i)_{i=1}^n$ *are the i.i.d. sample of* (Y, Z, W) *with* $\mathbb{E}|U|^{2+\delta} < \infty$ *and* $\mathbb{E}\|Z\|^{2+\delta} < \infty$ *for some* $\delta > 0$*; (ii)* $\mathbb{E}\|ZW\|^2 < \infty$, $\mathbb{E}\|UW\|^2 < \infty$, $\mathbb{E}\|UZW\| < \infty$, $\mathbb{E}\|Z\|^2\|W\| < \infty$, *and* $\mathbb{E}\left[|U|\|Z\|\|W\|^2\right] < \infty$*; and (iii)* $W^0 \neq 0$.

Assumption 4.2. *(i)* $\mu_1 \in \mathcal{R}(K^*K)^\gamma$ *and* $W \in \mathcal{R}(K^*K)^{\tilde{\gamma}}$ *for some* $\gamma, \tilde{\gamma} > 0$ *with* $\tilde{\gamma} + \gamma \geq 1/2$*; (ii)* $\alpha_n \to 0$, $n\alpha_n \to \infty$, $\pi_n \alpha_n^{(\gamma+\beta/2)\wedge 1} \to 0$, $\frac{\pi_n \alpha_n^{\gamma \wedge \frac{1}{2}}}{n\alpha_n} \to 0$, *and* $n\alpha_n^{1+\beta\wedge 1} \to 0$ *as* $n \to \infty$.

Assumption 4.2 is most restrictive when $\pi_n = O\left(n^{1/2}\right)$. In this case, we need $n\alpha_n^{(2\gamma+\beta)\wedge 2} \to 0$, $n\alpha_n^{2-2\gamma\wedge 1} \to \infty$, and $n\alpha_n^{1+\beta\wedge 1} \to 0$. These conditions are not binding whenever $\gamma \geq 1/2$ and $\beta > 0$.

Consider the sequence $\pi_n = n^{1/2}\left\|\Sigma^{1/2}K(\alpha_n I + K^*K)^{-1}\mu_1\right\|^{-1}$, where Σ is the variance operator of $(Y - \langle Z, \varphi_1\rangle)W$. Let $(\chi_j^2)_{j\geq 1}$ be i.i.d. chi-squared random variables with 1 degree of freedom and $(\lambda_j)_{j\geq 1}$ be the eigenvalues of $T : \phi \mapsto \mathbb{E}_X[\phi(X)h(X, X')]$ on $L_2(X)$ with[8]

$$h(X, X') = \frac{\langle W^0, W^{0\prime}\rangle}{2}\left\{\langle Z, \mu_0\rangle(Y' - \langle Z', \varphi_1\rangle) + \langle Z', \mu_0\rangle(Y - \langle Z, \varphi_1\rangle)\right\}.$$

The following result holds:

Theorem 4.1. *Suppose that Assumptions 3.1, 4.1, and 4.2 are satisfied. Then*

1. *If* $\pi_n = o(n\alpha_n)$, *then*
$$\pi_n\langle\hat{\varphi} - \varphi_1, \mu\rangle \xrightarrow{d} N(0, 1).$$

2. *If* $n\alpha_n = o(\pi_n)$, *then*
$$n\alpha_n\langle\hat{\varphi} - \varphi_1, \mu\rangle \xrightarrow{d} \left(\mathbb{E}\left[\|W\|^2(Y - \langle Z, \varphi_1\rangle)\langle Z, \mu_0\rangle\right] + \sum_{j\geq 1}\lambda_j(\chi_j^2 - 1)\right).$$

3. *If* $\pi_n = n\alpha_n$, *then*
$$n\alpha_n\langle\hat{\varphi} - \varphi_1, \mu\rangle \xrightarrow{d} N(0, 1) + \left(\mathbb{E}\left[\|W\|^2(Y - \langle Z, \varphi_1\rangle)\langle Z, \mu_0\rangle\right] + \sum_{j\geq 1}\lambda_j(\chi_j^2 - 1)\right).$$

[8] We use $\mathbb{E}_X$ to denote the expectation with respect to $X = (Y, Z, W)$ and X' is an independent copy of X.

The normalizing sequence in Theorem 4.1 can be π_n or $n\alpha_n$, and the asymptotic distribution may have a Gaussian component, a chi-squared component, or both, depending on the mapping properties of the operators K and Σ. More precisely, when the completeness condition is satisfied, the asymptotic distribution has only the Gaussian component. On the other hand, when the completeness condition fails, the chi-squared component may emerge. Lastly, in the extreme case of nonidentification when the instrumental variable is irrelevant, there is only the chi-squared component.

5 Monte Carlo Experiments

In this section, we study the validity of our asymptotic theory using Monte Carlo experiments. The DGP is

$$\varphi(z) = \sum_{j=1}^{\infty}(-1)^j j^{-6}\varphi_j(z), \qquad Y = \mathbb{E}[\varphi(Z)|W] + U, \qquad U \sim N(0, 0.01),$$

$$f_{ZW}^J(z,w) = C_J \sum_{j=1}^{J} j^{-3}\varphi_j(z)\varphi_j(w), \qquad \forall z, w \in [0,1];$$

see also Hall and Horowitz (2005). We use the following basis $\varphi_j(x) = \sqrt{2}\sin(j\pi x), j \geq 1$ and set C_J, so that the density integrates to 1. The parameter J controls the degree of identification. Note that the null space of the integral operator $K : L_2[0,1] \to L_2[0,1]$ with kernel f_{ZW}^J is infinite-dimensional whenever $J < \infty$. More precisely, K has J-dimensional range and

$$\mathcal{N}(K) \subset \operatorname{span}\{\varphi_j : \ j > J\}.$$

Therefore, we consider three cases with $J \in \{1, 2, \infty\}$, where $J = \infty$ corresponds to the identified model.

We simulate samples of size $n \in \{100, 500\}$ and use $5,000$ replications. For simplicity, we focus on the Tikhonov-regularized estimator with a product of Gaussian kernels to estimate the best approximation φ_1. The bandwidth parameter is selected using Silverman's rule of thumb, while the regularization parameter is chosen with leave-one-out cross-validation; see Babii (2020) and Centorrino (2014) for more details on the numerical implementation. All integrals are approximated with the Riemann sum on a uniform grid in $[0,1]$ and the infinite sums are truncated at some large finite value beyond which the results do not change numerically.

Table 1 displays the empirical L_2 and L_∞ errors for the three different degrees of identification measured by $J \in \{1, 2, \infty\}$. The estimation errors decrease when

	$n = 100$		$n = 500$	
J	L_2	L_∞	L_2	L_∞
1	0.0378	0.3295	0.0162	0.2156
2	0.0345	0.3199	0.0130	0.2054
∞	0.0311	0.3109	0.0107	0.1974

Table 1: L_2 and L_∞ errors. $5,000$ Monte Carlo experiments.

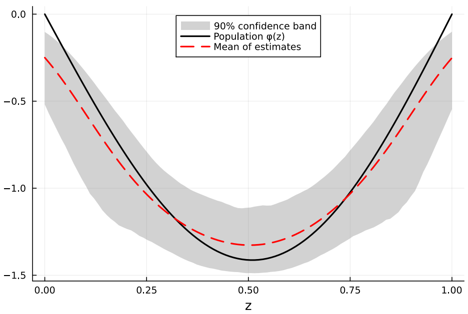
the dimension of the null space decreases, both when the sample size is $n = 100$ and $n = 500$. Larger samples also lead to smaller errors, which is in line with the asymptotic theory. Figure 1 plots the population value of the structural function, φ, the mean value of estimates $\mathbb{E}\hat{\varphi}$, and the pointwise 90% empirical confidence band computed over $5,000$ Monte Carlo replications. We can see the same pattern as in Table 1. Note that when $J = 1$, we can only recover the information related to the first basis vector, which is an example of extreme failure of the completeness condition. Nevertheless, even in this case, we can still learn a significant amount of information about the structural function, especially when the sample size is $n = 500$.

6 Conclusion

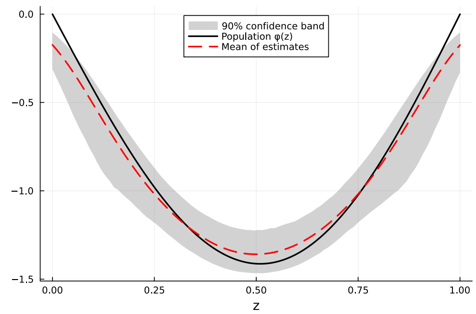
This paper develops a theory of nonidentified ill-posed inverse problems using the nonparametric IV and the high-dimensional regressions as illustrating examples. Identification failures occur due to the non-injectivity of the covariance or the conditional expectation operators. We show that if these operators are not injective, the estimators based on spectral regularization converge to the best approximation of the structural parameter in the orthogonal complement to the null space of the operator and derive new uniform and Hilbert space norm bounds for the risk.

This provides us an appealing projection interpretation for the nonparametric IV regression under identification failures similar to the one shared by the ordinary least-squares under misspecification. We describe several circumstances when the best approximation coincides with the structural parameter or at least reasonably approximates it and discuss how our results can be useful in the partial identification setting. It is also worth mentioning that the smoothness constraints can also be incorporated in the regularization procedure leading to more precise estimates in small samples; see Babii and Florens (2024) for regularization with a Sobolev-type penalty.

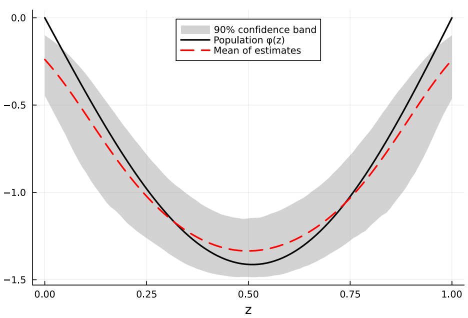
Lastly, we illustrate that under identification failures, the asymptotic distribution



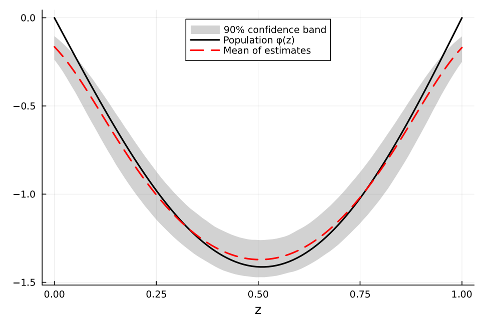
(a) $n = 100$ and $J = 1$



(b) $n = 500$ and $J = 1$



(c) $n = 100$ and $J = \infty$



(d) $n = 500$ and $J = \infty$

Figure 1: Mean estimates with pointwise 90% empirical confidence bands based on $5,000$ experiments.

of linear functionals can transition between the weighted chi-squared and Gaussian components.

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APPENDIX

A.1 Spectral Regularization

Consider a linear operator equation:

$$K\varphi = r,$$

where $K : \mathcal{E} \to \mathcal{H}$ is a linear operator and $\mathcal{E}$ and $\mathcal{H}$ are Hilbert spaces. The operator K is assumed to be bounded with $\|K\|^2 \leq \Lambda$ for some $\Lambda < \infty$, but not necessarily compact. Then $K^*K : \mathcal{E} \to \mathcal{E}$ is a normal operator with spectral decomposition:

$$K^*K = \int_{\sigma(K^*K)} \lambda \mathrm{d}E(\lambda),$$

where $\sigma(K^*K)$ is the spectrum of K^*K and E is the resolution of identity; see Rudin (1991), Theorem 12.23. For a bounded Borel function $g : [0, \Lambda] \to \mathbf{R}$, define:

$$g(K^*K) = \int_{\sigma(K^*K)} g(\lambda)\mathrm{d}E(\lambda).$$

If additionally the operator K is compact, the spectrum of K^*K is countable, and

$$g(K^*K) = \sum_{j=1}^{\infty} g(\lambda_j)E_j,$$

where E_j is a projection operator on the eigenspace corresponding to λ_j. If $(\varphi_j, \psi_j)_{j\geq 1}$ is a sequence of eigenvectors of K^*K, then for all $\varphi \in \mathcal{E}$

$$g(K^*K)\varphi = \sum_{j=1}^{\infty} g(\lambda_j)\langle\varphi, \varphi_j\rangle\psi_j.$$

We are interested in recovering the best approximation to the structural parameter φ when estimates $(\hat{K}, \hat{r})$ of (K, r) are available with $\|\hat{K}\|^2 \leq \Lambda$ almost surely. To that end, we consider a slightly more general version of Assumption 3.1:

Assumption A.1.1. *Suppose that (φ, K) belongs to*

$$\mathcal{F}(\beta, C) = \left\{(\varphi, K) : \ \varphi_1 = s_\beta(K^*K)\psi, \ \|\psi\|^2 \vee \|\varphi_0\| \vee \|K\| \leq C\right\},$$

where $s_\beta : [0, \Lambda] \to \mathbf{R}$ is a nondecreasing positive function such that $\lambda \mapsto s_\beta^2(\lambda)/\lambda^\beta$ is nonincreasing.

The following two cases are of interest:

1. Mildly ill-posed problem: $s_\beta(\lambda) = \lambda^{\beta/2}$.

2. Severely ill-posed problem: $s_\beta(\lambda) = \log^{-\beta/2}\left(\frac{1}{\lambda}\right)$ with $s_\beta(0) = 0$.

It is worth mentioning that the mildly ill-posed case allows for the exponential decline of eigenvalues of K^*K provided that the Fourier coefficients of φ_1 also decline exponentially fast. On the other hand, the severely ill-posed case allows for less regular φ_1.

The spectral regularization scheme is described by the family of bounded Borel functions $g_\alpha : [0,\infty) \to \mathbf{R}$, where $\alpha > 0$ is a regularization parameter such that $\lim_{\alpha\to 0} g_\alpha(\lambda) = \lambda^{-1}$. Assuming that $\hat{K}$ is bounded, the regularized estimator is defined as

$$\hat{\varphi} = g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{r}.$$

Our theoretical results require that the regularization scheme satisfies the following assumption:

Assumption A.1.2. *There exist* $c_1, c_2, c_3, \beta_0 > 0$ *such that: (i)* $\sup_{\lambda\leq\Lambda}|g_\alpha(\lambda)\lambda^{1/2}| \leq c_1/\sqrt{\alpha}$*; (ii)* $\sup_{\lambda\leq\Lambda}|(g_\alpha(\lambda)\lambda - 1)\lambda^r| \leq c_2\alpha^r, \forall r \in [0, 2\beta_0]$*; and (iii)* $\sup_{\lambda\leq\Lambda}|g_\alpha(\lambda)| \leq c_3/\alpha$*.*

It is easy to verify that the following regularization schemes satisfy Assumption A.1.2 in both the mildly and severely ill-posed cases:[9]

1. Tikhonov regularization:
$$g_\alpha(\lambda) = \frac{1}{\alpha+\lambda}.$$
Assumption A.1.2 holds with $c_1 = 1/2$, $c_2 = c_3 = 1$, and $\beta_0 = 2$.

2. Spectral cut-off regularization (principal components):
$$g_\alpha(\lambda) = \lambda^{-1}\mathbb{1}\{\lambda \geq \alpha\}.$$
Assumption A.1.2 holds with $c_1 = c_2 = c_3 = 1$ and every $\beta_0 > 0$.

[9] In the severely ill-posed case, the function $\lambda \mapsto s_\beta^2(\lambda)/\lambda^\beta$ is nonincreasing only on $(0, 1/e]$ and s_β is not defined at $\lambda = 1$. To address these issues, we can assume that the norm on $\mathcal{H}$ is scaled such that $\|K\|^2 \leq 1/e$.

3. Iterated Tikhonov regularization:

$$g_{\alpha,m}(\lambda) = \sum_{j=0}^{m-1} \frac{\alpha^j}{(\alpha+\lambda)^{j+1}} = \frac{1}{\lambda}\left(1 - \left(\frac{\alpha}{\lambda+\alpha}\right)^m\right), \qquad m = 2, 3, \ldots$$

Assumption A.1.2 holds with $c_1 = m^{1/2}$, $c_2 = 1$, and $c_3 = \beta_0 = m$.

4. Landweber-Fridman regularization:

$$g_{\alpha,c}(\lambda) = \sum_{j=0}^{1/\alpha-1} (1-c\lambda)^j = \frac{1}{\lambda}\left(1 - (1-c\lambda)^{1/\alpha}\right),$$

where $\alpha = 1/m$ for some $m \in \mathbf{N}$ and $c \in (0, 1/\Lambda)$. Assumption A.1.2 is satisfied with $c_1^2 = c$, $c_2 = c \vee 1$, $c_3 = \left(\frac{\beta}{ce}\right)^{\beta/2} \vee 1$, and every $\beta_0 > 0$.

The constant β_0 is called the qualification of the regularization scheme. It is well-known that simple Tikhonov regularization exhibits a saturation effect and its bias cannot converge faster than at the rate $O(\alpha^2)$. This limitation can be overcome with the iterated Tikhonov regularization similarly to using higher-order kernels for nonparametric kernel estimators.

The following result describes the convergence rate of $\hat{\varphi}$ to φ_1 for general regularization schemes:

Theorem A.1. *Suppose that Assumptions A.1.1, A.1.2, and 3.2 are satisfied with $\beta \leq \beta_0$.*[10] *Then in the mildly ill-posed case*

$$\|\hat{\varphi} - \varphi_1\|^2 = O_P\left(\frac{\delta_n}{\alpha_n} + \rho_{1n}^{\beta\wedge 1}\left(1 + \mathbb{1}_{\beta=1}\log^2 \rho_{1n}^{-1}\right) + \alpha_n^{\beta}\right),$$

while in the severely ill-posed case

$$\|\hat{\varphi} - \varphi_1\|^2 = O_P\left(\frac{\delta_n}{\alpha_n} + \log^{-\beta} \rho_{1n}^{-1} + \log^{-\beta} \alpha_n^{-1}\right).$$

Before we state the proof, note that

1. In the mildly ill-posed case, the optimal choice of regularization parameter is $\alpha_n \sim \delta_n^{\frac{1}{\beta+1}}$ provided that $\rho_{1n}^{\beta\wedge 1}\log^2 n \lesssim \delta_n/\alpha_n$. Then the convergence rate is

$$\|\hat{\varphi} - \varphi_1\|^2 = O_P\left(\delta_n^{\frac{\beta}{\beta+1}}\right).$$

[10] Note that we can always take $\beta = \beta_0$ if $\beta > \beta_0$.

2. In the severelly ill-posed case, one can choose $\alpha_n \sim \delta_n^{1/2}$. Then the convergence rate is
$$\|\hat{\varphi} - \varphi_1\|^2 = O_P\left(\frac{1}{\log^\beta n}\right),$$
provided that $\delta_n, \rho_n \sim n^{-c}$ for some $c > 0$.

Proof of Theorem A.1. Decompose

$$\hat{\varphi} - \varphi_1 = I_n + II_n + III_n$$

with
$$\begin{aligned}
I_n &= g_\alpha(\hat{K}^*\hat{K})\hat{K}^*(\hat{r} - \hat{K}\varphi), \\
II_n &= \left[g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I\right] s_\beta(\hat{K}^*\hat{K})\psi, \\
III_n &= \left[g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I\right] \left\{s_\beta(K^*K) - s_\beta(\hat{K}^*\hat{K})\right\}\psi.
\end{aligned}$$

To see that this decomposition holds, note that $\varphi = \varphi_1 + \varphi_0$ and that under Assumption A.1.1, $\varphi_1 = s_\beta(K^*K)\psi$. By the isometry of functional calculus

$$\begin{aligned}
\|I_n\|^2 &\leq \left\|g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\right\|^2 \left\|\hat{r} - \hat{K}\varphi_1\right\|^2 \\
&\leq \sup_{\lambda\leq\Lambda} \left|g_\alpha(\lambda)\lambda^{1/2}\right|^2 \left\|\hat{r} - \hat{K}\varphi_1\right\|^2,
\end{aligned}$$

which under Assumptions A.1.2 (i) and 3.2 shows that $\mathbb{E}\|I_n\|^2 = O\left(\delta_n/\alpha_n\right)$.

Next
$$\begin{aligned}
\|II_n\|^2 &\leq \left\|\left[g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I\right] s_\beta(\hat{K}^*\hat{K})\right\|^2 \|\psi\|^2 \\
&\leq \sup_{\lambda\leq\Lambda} \left|(g_\alpha(\lambda)\lambda - 1)s_\beta(\lambda)\right|^2 C.
\end{aligned}$$

Under Assumption A.1.1, s_β is nondecreasing, whence under Assumption A.1.2 (ii) with $r = 0$, we obtain

$$|(g_\alpha(\lambda)\lambda - 1)s_\beta(\lambda)| \leq c_2|s_\beta(\lambda)| \leq c_2 s_\beta(\alpha_n), \qquad \forall\lambda \in [0, \alpha_n].$$

Similarly, since $\lambda \mapsto s_\beta(\lambda)/\lambda^{\beta/2}$ is nonincreasing, under Assumption A.1.2 (ii) with $r = \beta/2$

$$|(g_\alpha(\lambda)\lambda - 1)s_\beta(\lambda)| \leq \left|(g_\alpha(\lambda)\lambda - 1)\lambda^{\beta/2}\right| \left|\frac{s_\beta(\lambda)}{\lambda^{\beta/2}}\right| \leq c_2\alpha_n^{\beta/2} s_\beta(\alpha_n)/\alpha_n^{\beta/2}, \qquad \forall\lambda \geq \alpha_n.$$

Therefore, $\|II_n\|^2 \leq Cc_2^2 s_\beta^2(\alpha_n)$.

Lastly, under Assumption A.1.2 (ii) with $r = 0$

$$
\begin{aligned}
\|III_n\|^2 &\leq \left\| g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I \right\|^2 \left\| s_\beta(\hat{K}^*\hat{K}) - s_\beta(K^*K) \right\|^2 \|\psi\|^2 \\
&\leq C \sup_{\lambda \leq \Lambda} |(g_\alpha(\lambda)\lambda - 1)|^2 \left\| s_\beta(\hat{K}^*\hat{K}) - s_\beta(K^*K) \right\|^2 \\
&\leq Cc_2^2 \left\| s_\beta(\hat{K}^*\hat{K}) - s_\beta(K^*K) \right\|^2 .
\end{aligned}
$$

The result follows by Lemma A.1.1 appearing at the end of the section. □

The following result provides the uniform convergence rates for a generic class of regularized estimators:

Theorem A.2. *Suppose that Assumptions A.1.1, A.1.2, 3.2, and 3.3 are satisfied with $\varphi_1 = s_\beta(K^*K)K^*\psi$ and $\beta_0 \geq \beta$. Then in the mildly ill-posed case*

$$
\|\hat{\varphi} - \varphi_1\|_\infty = O_P\left(\frac{\delta_n^{1/2}}{\alpha_n} + \frac{1}{\alpha_n^{1/2}} \left(\rho_{2n}^{1/2} + \rho_{1n}^{\frac{\beta \wedge 1}{2}} \left(1 + \mathbb{1}_{\beta=1} \log \rho_{1n}^{-1}\right) \right) + \alpha_n^{\beta/2} \right),
$$

while in the severely ill-posed case

$$
\|\hat{\varphi} - \varphi_1\|_\infty = O_P\left(\frac{\delta_n^{1/2}}{\alpha_n} + \frac{1}{\alpha_n^{1/2}} \left(\rho_{2n}^{1/2} + \log^{-\beta/2} \rho_{1n}^{-1} \right) + \log^{-\beta/2} \alpha_n^{-1} \right).
$$

Proof. Consider the following decomposition

$$
\hat{\varphi} - \varphi_1 = I_n + II_n + III_n
$$

with

$$
\begin{aligned}
I_n &= g_\alpha(\hat{K}^*\hat{K})\hat{K}^*(\hat{r} - \hat{K}\varphi_1), \\
II_n &= \left[g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I \right] s_\beta(\hat{K}^*\hat{K})\hat{K}^*\psi, \\
III_n &= \left[g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I \right] \left\{ s_\beta(K^*K)K^* - s_\beta(\hat{K}^*\hat{K})\hat{K}^* \right\} \psi.
\end{aligned}
$$

We bound the first term as

$$
\begin{aligned}
\|I_n\|_\infty &= \left\| \hat{K}^* g_\alpha(\hat{K}\hat{K}^*)(\hat{r} - \hat{K}\varphi_1) \right\| \\
&\leq \|\hat{K}^*\|_{2,\infty} \sup_{\lambda \leq \Lambda} |g_\alpha(\lambda)| \left\| \hat{r} - \hat{K}\varphi_1 \right\| \\
&= O_P\left(\frac{\delta_n^{1/2}}{\alpha_n} \right),
\end{aligned}
$$

where the last line follows under Assumptions 3.3 and A.1.2 (iii).

For the second term, note that

$$\begin{aligned}\|II_n\|_\infty &= \left\| \hat{K}^* \left[g_\alpha(\hat{K}\hat{K}^*)\hat{K}\hat{K}^* - I \right] s_\beta(\hat{K}\hat{K}^*)\psi \right\| \\ &\leq \|\hat{K}^*\|_{2,\infty} \sup_{\lambda \leq \Lambda} |(g_\alpha(\lambda)\lambda - 1)s_\beta(\lambda)| \|\psi\| \\ &= O_P(s_\beta(\alpha_n)),\end{aligned}$$

where the last line follows by the same argument as in the proof of Theorem A.1 under Assumptions 3.3 and A.1.2 (ii).

Next, for the third term,

$$\|III_n\|_\infty \leq \left\| g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I \right\|_\infty \left\| s_\beta(K^*K)K^* - s_\beta(\hat{K}^*\hat{K})\hat{K}^* \right\|_{2,\infty} \|\psi\|.$$

Under Assumptions A.1.2 (i) and 3.3

$$\begin{aligned}\left\| g_\alpha(\hat{K}^*\hat{K})\hat{K}^*\hat{K} - I \right\|_\infty &\leq \|\hat{K}^*\|_{2,\infty} \left\| g_\alpha(\hat{K}\hat{K}^*)\hat{K} \right\| + 1 \\ &\leq \|\hat{K}^*\|_{2,\infty} \sup_{\lambda \leq \Lambda} \left| g_\alpha(\lambda)\lambda^{1/2} \right| + 1 \\ &= O_P\left(\frac{1}{\alpha_n^{1/2}} \right).\end{aligned}$$

Lastly, under Assumption 3.3

$$\begin{aligned}\left\| s_\beta(K^*K)K^* - s_\beta(\hat{K}^*\hat{K})\hat{K}^* \right\|_{2,\infty} &= \left\| K^* s_\beta(KK^*) - \hat{K}^* s_\beta(\hat{K}\hat{K}^*) \right\|_{2,\infty} \\ &\lesssim \left\| \hat{K}^* - K^* \right\|_{2,\infty} + \left\| s_\beta(\hat{K}^*\hat{K}) - s_\beta(K^*K) \right\| \\ &\lesssim \rho_{2n}^{1/2} + \left\| s_\beta(\hat{K}^*\hat{K}) - s_\beta(K^*K) \right\|.\end{aligned}$$

The result now follows by Lemma A.1.1 as in the proof of Theorem A.1. $\square$

Lemma A.1.1. *Suppose that Assumption 3.2 (iii) is satisfied with $\rho_{1n} \sim n^{-c}$ for some $c > 0$. Then*

$$\left\| (\hat{K}^*\hat{K})^{\beta/2} - (K^*K)^{\beta/2} \right\| = O_P\left(\left(1 + \mathbb{1}_{\beta=1} \log \rho_{1n}^{-1} \right) \rho_{1n}^{\frac{\beta \wedge 1}{2}} \right)$$

and

$$\left\| \log^{-\beta/2}(\hat{K}^*\hat{K})^{-1} - \log^{-\beta/2}(K^*K)^{-1} \right\| = O_P\left(\log^{-\beta/2} \rho_{1n}^{-1} \right).$$

Proof. By Egger (2005), Lemma 3.2,

$$\left\|(\hat{K}^*\hat{K})^{\beta/2} - (K^*K)^{\beta/2}\right\| \lesssim \begin{cases} \|\hat{K} - K\|^{\beta}, & \beta < 1 \\ \|\hat{K} - K\| \left\{1 + \|\hat{K}\| + \|K\| + \log \|\hat{K} - K\|^{-1}\right\}, & \beta = 1 \\ \|\hat{K} - K\|(\|\hat{K}\| + \|K\|)^{\beta/2}, & \beta > 1. \end{cases}$$

Then the first statement follows since $\|\hat{K} - K\| = O_P(\rho_{1n}^{1/2})$ with $\rho_{1n} \to 0$ as $n \to \infty$ under Assumption 3.2 (iii) and since $x \mapsto x \log(1/x)$ is strictly increasing in the neighborhood of zero.

For the second statement, by Mathé and Pereverzev (2002), Theorem 4

$$\begin{aligned} \left\|\log^{-\beta/2}(\hat{K}^*\hat{K})^{-1} - \log^{-\beta/2}(K^*K)^{-1}\right\| &= O_P\left(\log^{-\beta/2} \|\hat{K}^*\hat{K} - K^*K\|^{-1}\right) \\ &= O_P\left(\log^{-\beta/2} \rho_{1n}^{-1}\right), \end{aligned}$$

where the second line follows under Assumption 3.2 (iii) and since $x \mapsto \log^{-\beta/2}(1/x)$ is strictly increasing in the neighborhood of zero. □

A.2 Proofs of Main Results

Proof of Theorem 3.1. Decompose

$$\begin{aligned} \hat{\varphi} - \varphi_1 &= (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*(\hat{r} - \hat{K}\varphi_1) \\ &\quad + (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\hat{K}\varphi_1 - (\alpha_n I + K^*K)^{-1}K^*K\varphi_1 \\ &\quad + \left((\alpha_n I + K^*K)^{-1}K^*K - I\right)\varphi_1 \\ &\triangleq I_n + II_n + III_n. \end{aligned}$$

III_n is the regularization bias that can controlled under Assumption 3.1

$$\begin{aligned} \|III_n\|^2 &= \left\|\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1\right\|^2 \\ &\leq C\left\|\alpha_n(\alpha_n I + K^*K)^{-1}(K^*K)^{\beta/2}\right\|^2 \\ &\leq C \sup_{\lambda \in [0, \|K\|^2]} \left|\frac{\alpha_n \lambda^{\beta/2}}{\alpha_n + \lambda}\right|^2 \\ &\leq C^{(2\beta-3)\vee 1}\alpha_n^{\beta}, \end{aligned}$$

see Babii (2022). The first term is controlled under Assumption 3.2 (i)

$$
\begin{aligned}
\mathbb{E}\|I_n\|^2 &\leq \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\|^2 \left\|\hat{r} - \hat{K}\varphi_1\right\|^2 \\
&\leq \sup_{\lambda \geq 0}\left|\frac{\lambda^{1/2}}{\alpha_n + \lambda}\right|^2 \mathbb{E}\left\|\hat{r} - \hat{K}\varphi_1\right\|^2 \\
&\leq \frac{1}{4\alpha_n}\mathbb{E}\left\|\hat{r} - \hat{K}\varphi_1\right\|^2 \\
&\leq \frac{C_1\delta_n}{4\alpha_n}.
\end{aligned}
$$

The second term is decomposed further

$$
\begin{aligned}
II_n &= -\left[\alpha_n(\alpha_n I + \hat{K}^*\hat{K})^{-1} - \alpha_n(\alpha_n I + K^*K)^{-1}\right]\varphi_1 \\
&= -(\alpha_n I + \hat{K}^*\hat{K})^{-1}\alpha_n\left[K^*K - \hat{K}^*\hat{K}\right](\alpha_n I + K^*K)^{-1}\varphi_1 \\
&= (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\left[\hat{K} - K\right]\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1 \\
&\quad + (\alpha_n I + \hat{K}^*\hat{K})^{-1}\left[\hat{K}^* - K^*\right]\alpha_n K(\alpha_n I + K^*K)^{-1}\varphi_1 \\
&= II_n^a + II_n^b.
\end{aligned}
$$

It follows from the previous computations and Assumption 3.2 (iii) that

$$
\begin{aligned}
\mathbb{E}\|II_n^a\|^2 &= \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\left[\hat{K} - K\right]\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1\right\|^2 \\
&\leq \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\|^2 \left\|\hat{K} - K\right\|^2 \left\|\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1\right\|^2 \\
&\leq \sup_{\lambda \geq 0}\left|\frac{\lambda^{1/2}}{\alpha_n + \lambda}\right|^2 \mathbb{E}\left\|\hat{K} - K\right\|^2 C^{(2\beta-3)\vee 1}\alpha_n^{\beta\wedge 2} \\
&\leq \frac{C_2\rho_{1n}}{4\alpha_n}C^{(2\beta-3)\vee 1}\alpha_n^{\beta\wedge 2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\|II_n^b\|^2 &= \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\left[\hat{K}^* - K^*\right]\alpha_n K(\alpha_n I + K^*K)^{-1}\varphi_1\right\|^2 \\
&\leq \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\|^2 \left\|\hat{K}^* - K^*\right\|^2 C\left\|\alpha_n K(\alpha_n I + K^*K)^{-1}(K^*K)^{\beta/2}\right\|^2 \\
&\leq \sup_{\lambda\geq 0}\left|\frac{1}{\alpha_n + \lambda}\right|^2 \mathbb{E}\left\|\hat{K}^* - K^*\right\|^2 C \sup_{\lambda\in[0,C^2]}\left|\frac{\alpha_n\lambda^{(\beta+1)/2}}{\alpha_n + \lambda}\right|^2 \\
&\leq \frac{C_2\rho_{1n}}{\alpha_n^2} C^{(2\beta-1)\vee 1}\alpha_n^{(\beta+1)\wedge 2},
\end{aligned}
$$

where we use $\|\hat{K}^* - K^*\| = \|\hat{K} - K\|$. Combining all estimates together, we obtain the result. □

Proof of the Theorem 3.2. Consider the same decomposition as in the proof of Theorem 3.1. Since $\varphi_1 = (K^*K)^{\beta/2}K^*\psi$, the bias term is treated similarly to the identified case, see Babii (2020), Proposition 3.1

$$
\begin{aligned}
\|III_n\|_\infty &= \left\|\alpha_n K^*(\alpha_n I + KK^*)^{-1}(KK^*)^{\frac{\beta}{2}}\psi\right\|_\infty \\
&\leq \|K^*\|_{2,\infty}\left\|\alpha_n(\alpha_n I + KK^*)^{-1}(KK^*)^{\frac{\beta}{2}}\right\|\|\psi\| \\
&= O(\alpha_n^{\beta/2}).
\end{aligned}
$$

Next, by the Cauchy-Schwartz inequality and Assumption 3.2 (iii) and Assumption 3.3, the first term is

$$
\begin{aligned}
\mathbb{E}\|I_n\|_\infty &= \mathbb{E}\left\|\hat{K}^*(\alpha_n I + \hat{K}\hat{K}^*)^{-1}(\hat{r} - \hat{K}\varphi_1)\right\|_\infty \\
&\leq \mathbb{E}\|\hat{K}^*\|_{2,\infty}\left\|(\alpha_n I + \hat{K}\hat{K}^*)^{-1}\right\|\left\|(\hat{r} - \hat{K}\varphi_1)\right\| \\
&\leq \frac{1}{\alpha_n}\left(\|K^*\|_{2,\infty}\mathbb{E}\left\|(\hat{r} - \hat{K}\varphi_1)\right\| + \mathbb{E}\left\|\hat{K}^* - K^*\right\|_{2,\infty}\left\|(\hat{r} - \hat{K}\varphi_1)\right\|\right) \\
&\leq C_1^{1/2}\left(C_3 + C_3^{1/2}\rho_{2n}^{1/2}\right)\frac{\delta_n^{1/2}}{\alpha_n}.
\end{aligned}
$$

The second term is decomposed further similarly as in the proof of Theorem 3.1

in II_n^a and II_n^b. We bound each of the two terms separately. First,

$$
\begin{aligned}
\mathbb{E}\|II_n^a\|_\infty &= \left\|\hat{K}^*(\alpha_n I + \hat{K}\hat{K}^*)^{-1}\left[\hat{K} - K\right]\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1\right\|_\infty \\
&\leq \mathbb{E}\left\|\hat{K}^*\right\|_{2,\infty}\left\|(\alpha_n I + \hat{K}\hat{K}^*)^{-1}\right\|\left\|\hat{K} - K\right\|\left\|\alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1\right\| \\
&\leq \frac{1}{\alpha_n}\left(C_3 + \left(\mathbb{E}\left\|\hat{K}^* - K^*\right\|_{2,\infty}^2\right)^{1/2}\right)\left(\mathbb{E}\left\|\hat{K} - K\right\|^2\right)^{1/2} C^{(\beta-1.5)\vee 0.5}\alpha_n^{\beta/2} \\
&\leq \left(C_3 + C_3^{1/2}\rho_{2n}^{1/2}\right)\frac{C_2^{1/2}\rho_{1n}^{1/2}}{\alpha_n} C^{(\beta-1.5)\vee 0.5}\alpha_n^{\beta/2}.
\end{aligned}
$$

Second, under Assumption 3.3, by the inequality in Babii (2020), Lemma A.4.1, see also Nair (2009), Problem 5.8

$$
\begin{aligned}
\mathbb{E}\|II_n^b\|_\infty &= \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\left[\hat{K}^* - K^*\right]\alpha_n K(\alpha_n I + K^*K)^{-1}\varphi_1\right\|_\infty \\
&= \mathbb{E}\left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\|_\infty\left\|\hat{K}^* - K^*\right\|_{2,\infty}\left\|\alpha_n K(\alpha_n I + K^*K)^{-1}\varphi_1\right\| \\
&= \frac{1}{2\alpha_n^{3/2}}\mathbb{E}\left(\|\hat{K}^*\|_{2,\infty} + 2\alpha_n^{1/2}\right)\left\|\hat{K}^* - K^*\right\|_{2,\infty} C^{(\beta-1/2)\vee 1/2}\alpha_n^{\frac{\beta+1}{2}\wedge 1} \\
&\leq \frac{1}{2\alpha_n^{3/2}}\left(C_3 + C_3^{1/2}\rho_{2n}^{1/2} + 2\alpha_n^{1/2}\right) C_3^{1/2}\rho_{2n}^{1/2} C^{(\beta-1/2)\vee 1/2}\alpha_n^{\frac{\beta+1}{2}\wedge 1}.
\end{aligned}
$$

Collecting all estimates together, we obtain the result. □

The following proposition provides low-level conditions for Assumptions 3.2 and 3.3 in the nonparametric IV regression estimated with kernel smoothing. Let C_M^s denote the the Hölder class.

Proposition A.2.1. *Suppose that (i) $(Y_i, Z_i, W_i)_{i=1}^n$ are i.i.d. and $\mathbb{E}|Y_1|^2 \leq C\infty$; (ii) $f_{ZW} \in C_M^s$; (iii) kernel functions $K_z : \mathbf{R}^p \to \mathbf{R}$ and $K_w : \mathbf{R}^q \to \mathbf{R}$ are such that for $l \in \{w, z\}$, $K_l \in L_1 \cap L_2$, $\int K_l(u)\mathrm{d}u = 1$, $\int \|u\|^s K_l(u)\mathrm{d}u < \infty$, and $\int u^k K_l(u)\mathrm{d}u = 0$ for all multindices $|k| = 1, \dots, \lfloor s \rfloor$. Then*

$$
\mathbb{E}\left\|\hat{r} - \hat{K}\varphi_1\right\|^2 = O\left(\frac{1}{nh_n^q} + h_n^{2s}\right) \qquad \text{and} \qquad \mathbb{E}\left\|\hat{K} - K\right\|^2 = O\left(\frac{1}{nh_n^{p+q}} + h_n^{2s}\right),
$$

where the constants do not depend on (K, φ).

Proof. For the first claim, note that

$$
\mathbb{E}\left\|\hat{r} - \hat{K}\varphi_1\right\|^2 \leq 2\mathbb{E}\left\|\hat{r} - r\right\|^2 + 2\mathbb{E}\left\|(\hat{K} - K)\varphi_1\right\|^2.
$$

Decompose

$$\mathbb{E}\left\|\hat{r}-r\right\|^2=\mathbb{E}\left\|\hat{r}-\mathbb{E}\hat{r}\right\|^2+\left\|\mathbb{E}\hat{r}-r\right\|^2.$$

Under the i.i.d. assumption

$$\begin{aligned}
\mathbb{E}\left\|\hat{r}-\mathbb{E}\hat{r}\right\|^2 &= \mathbb{E}\left\|\frac{1}{nh_n^q}\sum_{i=1}^n Y_i K_w\left(h_n^{-1}(W_i-w)\right)-\mathbb{E}\left[Y_i h_n^{-q}K_w\left(h_n^{-1}(W_i-w)\right)\right]\right\|^2 \\
&= \frac{1}{n}\mathbb{E}\left\|Y_i h_n^{-q}K_w\left(h_n^{-1}(W_i-w)\right)-\mathbb{E}\left[Y_i h_n^{-q}K_w\left(h_n^{-1}(W_i-w)\right)\right]\right\|^2 \\
&\leq \frac{1}{nh_n^q}\mathbb{E}|Y_1|^2\|K_w\|^2 \\
&= O\left(\frac{1}{nh_n^q}\right).
\end{aligned}$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}
\mathbb{E}\hat{r}-r &= \mathbb{E}\left[\varphi(Z_i)h_n^{-q}K_w\left(h_n^{-1}(W_i-w)\right)\right]-\int\varphi(z)f_{ZW}(z,w)\mathrm{d}z \\
&= \int\varphi(z)\left\{[f_{ZW}*K_w](z,w)-f_{ZW}(z,w)\right\}\mathrm{d}z \\
&\leq \|\varphi\|\left\|f_{ZW}*K_w-f_{ZW}\right\|,
\end{aligned}$$

where we put $[f_{ZW}*K_{w,h}](z,w)=\int f_{ZW}(z,w')h_n^{-q}K_w\left(h_n^{-1}(w-w')\right)\mathrm{d}w'$. Since $f_{ZW}\in C_M^s$, we obtain

$$\|\mathbb{E}\hat{r}-r\|=O(h^s),$$

see, e.g., Giné and Nickl (2016), Proposition 4.3.8. Therefore,

$$\mathbb{E}\|\hat{r}-r\|^2=O\left(\frac{1}{nh_n^q}+h_n^{2s}\right).$$

Next, decompose

$$(\hat{K}\varphi_1-K\varphi_1)(w)\triangleq V_n(w)+B_n(w)$$

with

$$\begin{aligned}
V_n &= \int\varphi_1(z)\left(\hat{f}_{ZW}(z,w)-\mathbb{E}\hat{f}_{ZW}(z,w)\right)\mathrm{d}z, \\
B_n &= \int\varphi_1(z)\left(\mathbb{E}\hat{f}_{ZW}(z,w)-f_{ZW}(z,w)\right)\mathrm{d}z.
\end{aligned}$$

By the Cauchy-Schwartz inequality

$$\|B_n\|\leq\|\varphi_1\|\left\|\mathbb{E}\hat{f}_{ZW}-f_{ZW}\right\|,$$

where the right side is of order $O(h_n^s)$ under the assumption $f_{ZW} \in C_M^s$, see Giné and Nickl (2016), p.404.

Next, note that

$$V_n(w) = \frac{1}{nh_n^q} \sum_{i=1}^{n} \eta_{n,i}(w).$$

with

$$\eta_{n,i}(w) = K_w\left(h_n^{-1}(W_i - w)\right)\left[\varphi_1 * K_z\right](Z_i) - \mathbb{E}\left[K\left(h_n^{-1}(W_i - w)\right)\left[\varphi_1 * K_z\right](Z_i)\right],$$

where $\left[\varphi_1 * K_z\right](Z_i) = \int \varphi_1(z) h_n^{-p} K_z\left(h_n^{-1}(Z_i - z)\right) \mathrm{d}z$. Then

$$\begin{aligned}
\mathbb{E}\|V_n\|^2 &\leq \frac{1}{nh_n^{2q}} \int\int\int \left|K_w(h_n^{-1}(w' - w))\right|^2 \left|\left[\varphi_1 * K_z\right](z')\right|^2 \mathrm{d}w f_{ZW}(z', w')\mathrm{d}w'\mathrm{d}z' \\
&= \frac{1}{nh_n^q}\|K_w\|^2 \int \left|\left[\varphi_1 * K_z\right](z)\right|^2 f_Z(z)\mathrm{d}z \\
&= O\left(\frac{1}{nh_n^q}\right),
\end{aligned}$$

where the second line follows by change of variables, and the last by $\|f_Z\|_\infty \leq C$, and Young's inequality. Combining all estimates together, we obtain the first claim.

The second claim follows from the inequality

$$\mathbb{E}\left\|\hat{K} - K\right\|^2 \leq \mathbb{E}\left\|\hat{f}_{ZW} - f_{ZW}\right\|^2$$

and the standard resultson the L_2 error of the kernel density estimator, Giné and Nickl (2016), Chapter 5. □

Proof of Theorem 4.1. Decompose

$$\langle\hat{\varphi} - \varphi_1, \mu\rangle = \langle\hat{\varphi} - \varphi_1, \mu_1\rangle + \langle\hat{\varphi} - \varphi_1, \mu_0\rangle.$$

We will show that

$$n\alpha_n\langle\hat{\varphi} - \varphi_1, \mu_0\rangle \xrightarrow{d} \mathbb{E}\left[\|W\|^2(Y - \langle Z, \varphi_1\rangle)\langle Z, \mu_0\rangle\right] + \sum_{j\geq 1}\lambda_j(\chi_j^2 - 1)$$

and that

$$\pi_n\langle\hat{\varphi} - \varphi, \mu_1\rangle \xrightarrow{d} N(0,1).$$

The final result follows from combining these two results together.

We fill focus on the distribution of $n\alpha_n\langle\hat{\varphi}-\varphi_1,\mu_0\rangle$ first. Put $b_n = \alpha_n(\alpha_n I + K^*K)^{-1}\varphi_1$ and note that $(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^* = \hat{K}^*(\alpha_n I + \hat{K}\hat{K}^*)^{-1}$. Then, similarly to the proof of Theorem 3.1, decompose

$$\begin{aligned}
\langle\hat{\varphi}-\varphi_1,\mu_0\rangle &= \left\langle \hat{K}^*(\alpha_n I + KK^*)^{-1}(\hat{r}-\hat{K}\varphi_1),\mu_0\right\rangle \\
&\quad + \left\langle \hat{K}^*\left((\alpha_n I + \hat{K}\hat{K}^*)^{-1} - (\alpha_n I + KK^*)^{-1}\right)(\hat{r}-\hat{K}\varphi_1),\mu_0\right\rangle \\
&\quad + \left\langle (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*(\hat{K}-K)b_n,\mu_0\right\rangle \\
&\quad + \left\langle (\alpha_n I + \hat{K}^*\hat{K})^{-1}(\hat{K}^*-K^*)Kb_n,\mu_0\right\rangle \\
&\quad + \langle b_n,\mu_0\rangle \\
&\triangleq I_n + II_n + III_n + IV_n + V_n.
\end{aligned}$$

We will show that

$$I_n = \left\langle \alpha_n(\alpha_n I + KK^*)^{-1}(\hat{r}-\hat{K}\varphi_1),\hat{K}\mu_0\right\rangle$$

is the leading term in this decomposition and that all other terms are asymptotically negligible. Note that since $\mu_0\in\mathcal{N}(K)$,

$$(\alpha_n I + K^*K)^{-1}\mu_0 = \frac{1}{\alpha_n}\mu_0. \tag{A.1}$$

Then

$$\begin{aligned}
II_n &= \left\langle (\alpha_n I + \hat{K}\hat{K}^*)^{-1}(KK^*-\hat{K}\hat{K}^*)(\alpha_n I + KK^*)^{-1}(\hat{r}-\hat{K}\varphi_1),\hat{K}\mu_0\right\rangle \\
&= \left\langle (\alpha_n I + \hat{K}\hat{K}^*)^{-1}\hat{K}(K^*-\hat{K}^*)(\alpha_n I + KK^*)^{-1}(\hat{r}-\hat{K}\varphi_1),\hat{K}\mu_0\right\rangle \\
&\quad + \left\langle (\alpha_n I + \hat{K}\hat{K}^*)^{-1}(K-\hat{K})K^*(\alpha_n I + KK^*)^{-1}(\hat{r}-\hat{K}\varphi_1),\hat{K}\mu_0\right\rangle.
\end{aligned}$$

By the Cauchy-Schwartz inequality and computations similar to those in the proof of Theorem 3.1

$$\begin{aligned}
II_n &\leq \left\|(\alpha_n I + \hat{K}\hat{K}^*)^{-1}\hat{K}\right\|\left\|K^*-\hat{K}^*\right\|\|(\alpha_n I + KK^*)^{-1}\|\left\|\hat{r}-\hat{K}\varphi_1\right\|\left\|\hat{K}\mu_0\right\| \\
&\quad + \left\|(\alpha_n I + \hat{K}\hat{K}^*)^{-1}\right\|\left\|K-\hat{K}\right\|\|K^*(\alpha_n I + KK^*)^{-1}\|\left\|\hat{r}-\hat{K}\varphi_1\right\|\left\|(\hat{K}-K)\mu_0\right\| \\
&\leq \frac{1}{\alpha_n^{3/2}}\left\|\hat{K}-K\right\|^2\left\|\hat{r}-\hat{K}\varphi_1\right\|\|\mu_0\|.
\end{aligned}$$

Next, under Assumption 3.1 from the proof of Theorem 3.1 we also know that $\|b_n\| = O(\alpha_n^{(\beta/2)\wedge 1})$ and that $\|Kb_n\| = O(\alpha_n^{((\beta+1)/2)\wedge 1})$. Therefore

$$\begin{aligned} III_n &\leq \left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\| \left\|\hat{K} - K\right\| \|b_n\|\|\mu_0\| \\ &\lesssim \frac{1}{\alpha_n^{1/2}} \left\|\hat{K} - K\right\| \alpha_n^{\frac{\beta}{2}\wedge 1} \end{aligned}$$

and

$$\begin{aligned} IV_n &\leq \left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\| \left\|\hat{K}^* - K^*\right\| \|Kb_n\|\|\mu_0\| \\ &\lesssim \frac{1}{\alpha_n^{1/2}} \left\|\hat{K} - K\right\| \alpha_n^{\frac{\beta\wedge 1}{2}}. \end{aligned}$$

Lastly, the bias is zero by equation (A.1) and the orthogonality between φ_1 and μ_0

$$\langle b_n, \mu_0\rangle = \left\langle \varphi_1, \alpha_n(\alpha_n I + K^*K)^{-1}\mu_0\right\rangle = \langle \varphi_1, \mu_0\rangle = 0.$$

It follows from the discussion in Section 3 that under Assumption 4.1

$$\left\|\hat{K} - K\right\| = O_P\left(\frac{1}{n^{1/2}}\right) \quad \text{and} \quad \left\|\hat{r} - \hat{K}\varphi_1\right\| = O_P\left(\frac{1}{n^{1/2}}\right).$$

Therefore, since under Assumption 4.2, $n\alpha_n^{1+\beta\wedge 1} \to 0$ and $n\alpha_n \to \infty$,

$$\begin{aligned} n\alpha_n\langle\hat{\varphi} - \varphi_1, \mu_0\rangle &= n\alpha_n \left\langle (\alpha_n I + KK^*)^{-1}(\hat{r} - \hat{K}\varphi_1), \hat{K}\mu_0 \right\rangle + o_P(1) \\ &\triangleq S_n + o_P(1) \end{aligned}$$

with

$$S_n = n\alpha_n \left\langle (\alpha_n I + KK^*)^{-1}(\hat{r} - \hat{K}\varphi_1), \hat{K}\mu_0 \right\rangle.$$

Next, decompose $S_n = S_n^0 + S_n^1$ with

$$\begin{aligned} S_n^0 &= \frac{1}{n}\sum_{i,j=1}^{n} (Y_i - \langle Z_i, \varphi_1\rangle) \langle Z_j, \mu_0\rangle \left\langle \alpha_n(\alpha_n I + KK^*)^{-1}W_i^0, W_j^0\right\rangle, \\ S_n^1 &= \frac{1}{n}\sum_{i,j=1}^{n} (Y_i - \langle Z_i, \varphi_1\rangle)\langle Z_j, \mu_0\rangle \left\langle \alpha_n(\alpha_n I + KK^*)^{-1}W_i^1, W_j^1\right\rangle. \end{aligned}$$

Since $W_i^0 \in \mathcal{N}(K^*)$, we have $(\alpha_n I + KK^*)^{-1} W_i^0 = \frac{1}{\alpha_n} W_i^0$. Using this fact, decompose further $S_n^0 \triangleq \zeta_n^0 + \mathbf{U}_n^0$ with

$$\zeta_n^0 = \frac{1}{n} \sum_{i=1}^n (Y_i - \langle Z_i, \varphi_1 \rangle) \langle Z_i, \mu_0 \rangle \left\| W_i^0 \right\|^2,$$

$$\mathbf{U}_n^0 = \frac{1}{n} \sum_{i<j} \left\{ \langle Z_i, \mu_0 \rangle (Y_j - \langle Z_j, \varphi_0 \rangle) + \langle Z_j, \mu_0 \rangle (Y_i - \langle Z_i, \mu_0 \rangle) \right\} \left\langle W_i^0, W_j^0 \right\rangle.$$

Under Assumption 4.1 by the strong law of large numbers

$$\zeta_n^0 \xrightarrow{a.s.} \mathbb{E} \left[\left\| W^0 \right\|^2 (Y - \langle Z, \varphi_1 \rangle) \langle Z, \mu_0 \rangle \right].$$

Next, note that $W^0 = P_0 W$ and $W^1 = (I - P_0) W$, where P_0 is the projection operator on $\mathcal{N}(K^*)$. Since projection is a bounded linear operator, it commutes with the expectation, cf., Bosq (2000), p.29, whence $\mathbb{E}\left[W^0 \langle Z, \mu_0 \rangle\right] = P_0 K \mu_0 = 0$ and $\mathbb{E}\left[W^0 (Y - \langle Z, \varphi_1 \rangle)\right] = P_0 \mathbb{E}[WU] + P_0 K \varphi_0 = 0$. Therefore, $\mathbf{U}_n^0$ is a centered degenerate U-statistics with the kernel function h. Under Assumption 4.1 by the CLT for the degenerate U-statistics, see Gregory (1977),

$$\mathbf{U}_n^0 \xrightarrow{d} \sum_{j=1}^\infty \lambda_j (\chi_j^2 - 1).$$

It remains to show that $S_n^1 = o_P(1)$. To that end decompose $S_n^1 = \zeta_n^1 + \mathbf{U}_n^1$ with

$$\zeta_n^1 = \frac{1}{n} \sum_{i=1}^n (Y_i - \langle Z_i, \varphi_1 \rangle) \langle Z_i, \mu_0 \rangle \left\langle \alpha_n (\alpha_n I + KK^*)^{-1} W_i^1, W_i^1 \right\rangle,$$

$$\mathbf{U}_n^1 = \frac{1}{n} \sum_{i \neq j} (Y_i - \langle Z_i, \varphi_1 \rangle) \langle Z_j, \mu_0 \rangle \left\langle \alpha_n (\alpha_n I + KK^*)^{-1} W_i^1, W_j^1 \right\rangle.$$

It follows from Bakushinskii (1967) that $\| \alpha_n (\alpha_n I + KK^*)^{-1} W^1 \| = o(1)$. Then under Assumption 4.1 by the dominated convergence theorem

$$\begin{aligned}
\mathbb{E} \left| \zeta_n^1 \right| &\leq \| \mu_0 \| \mathbb{E} \left[(|U| + \|Z\| \|\varphi_0\|) \|Z\| \left\| \alpha_n (\alpha_n I + KK^*)^{-1} W^1 \right\| \right] \\
&\lesssim \mathbb{E} \left[(\|UZ\| + \|Z\|^2) \|W\| \left\| \alpha_n (\alpha_n I + KK^*)^{-1} W^1 \right\| \right] \\
&= o(1),
\end{aligned}$$

whence by Markov's inequality $\zeta_n^1 = o_P(1)$. Lastly, note that

$$\begin{aligned}
\mathbf{U}_n^1 = \frac{1}{n} \sum_{i<j} \Big\{ & (Y_i - \langle Z_i, \varphi_1 \rangle) \langle Z_j, \mu_0 \rangle \left\langle \alpha_n (\alpha_n I + KK^*)^{-1} W_i^1, W_j^1 \right\rangle \\
& + (Y_j - \langle Z_j, \varphi_1 \rangle) \langle Z_i, \mu_0 \rangle \left\langle \alpha_n (\alpha_n I + KK^*)^{-1} W_j^1, W_i^1 \right\rangle \Big\}
\end{aligned}$$

is a centered degenerate U-statistics. Then by the moment inequality in Korolyuk and Borovskich (1994), Theorem 2.1.3,

$$
\begin{aligned}
\mathbb{E}\left|\mathbf{U}_n^1\right|^2 &\leq 2^{-1}\mathbb{E}\left|(U_1+\langle Z_1,\varphi_0\rangle)\langle Z_2,\mu_0\rangle\left\langle \alpha_n(\alpha_n I+KK^*)^{-1}W_1^1, W_2^1\right\rangle\right|^2 \\
&\quad + 2^{-1}\mathbb{E}\left|(U_2+\langle Z_2,\varphi_0\rangle)\langle Z_1,\mu_0\rangle\left\langle \alpha_n(\alpha_n I+KK^*)^{-1}W_2^1, W_1^1\right\rangle\right|^2 \\
&\lesssim \mathbb{E}\left[\|Z\|^2\left\|\alpha_n(\alpha_n I+KK^*)^{-1}W^1\right\|^2\right] = o(1),
\end{aligned}
$$

where the last line follows under Assumptions 4.1 and 4.2, and previous discussions. This shows that

$$
n\alpha_n\langle\hat{\varphi}-\varphi_1,\mu_0\rangle \xrightarrow{d} \mathbb{E}\left[\|W\|^2(Y-\langle Z,\varphi_1\rangle)\langle Z,\mu_0\rangle\right] + \sum_{j\geq 1}\lambda_j(\chi_j^2-1).
$$

Next, we focus on the distribution of $\pi_n\langle\hat{\varphi}-\varphi_1,\mu_1\rangle$. Decompose

$$
\begin{aligned}
\langle\hat{\varphi}-\varphi_1,\mu_1\rangle &= \left\langle(\alpha_n I+K^*K)^{-1}K^*(\hat{r}-\hat{K}\varphi_1),\mu_1\right\rangle \\
&\quad + \left\langle\left\{(\alpha_n I+\hat{K}^*\hat{K})^{-1}-(\alpha_n I+K^*K)^{-1}\right\}\hat{K}^*(\hat{r}-\hat{K}\varphi_1),\mu_1\right\rangle \\
&\quad + \left\langle(\alpha_n I+K^*K)^{-1}(\hat{K}^*-K^*)(\hat{r}-\hat{K}\varphi_1),\mu_1\right\rangle \\
&\quad + \left\langle(\alpha_n I+\hat{K}^*\hat{K})^{-1}\hat{K}^*\hat{K}\varphi_1-(\alpha_n I+K^*K)^{-1}K^*K\varphi_1,\mu_1\right\rangle \\
&\quad + \langle b_n,\mu_1\rangle \\
&\triangleq I_n'+II_n'+III_n'+IV_n'+\langle b_n,\mu_1\rangle.
\end{aligned}
$$

Put $\eta_n=(Y-\langle Z,\varphi_1\rangle)(\alpha_n I+K^*K)^{-1}K^*W$ and note that

$$
\begin{aligned}
\mathrm{Var}(\langle\eta_n,\mu_1\rangle) &= \mathbb{E}\left|\langle(Y-\langle Z,\varphi_1)W,K(\alpha_n I+K^*K)^{-1}\mu_1\rangle\right|^2 \\
&= \langle\Sigma K(\alpha_n I+K^*K)^{-1}\mu_1,K(\alpha_n I+K^*K)^{-1}\mu_1\rangle \\
&= \|\Sigma^{1/2}K(\alpha_n I+K^*K)^{-1}\mu_1\|^2.
\end{aligned}
$$

Suppose that for all $\epsilon>0$, the following Lindeberg's condition is satisfied

$$
\lim_{n\to\infty}\frac{\pi_n^2}{n}\mathbb{E}\left[|\langle\eta_n,\mu_1\rangle|^2\mathbb{1}_{\{\pi_n|\langle\eta_n,\mu_1\rangle|\geq\epsilon n\}}\right]=0 \tag{A.2}
$$

with $\pi_n=n^{1/2}\left\|\Sigma^{1/2}K(\alpha_n I+K^*K)^{-1}\mu_1\right\|^{-1}$. Then by the Lindeberg-Feller central limit theorem

$$
\begin{aligned}
\pi_n I_n' &= \frac{\pi_n}{n}\sum_{i=1}^n(U_i+\langle Z_i,\varphi_0\rangle)\left\langle(\alpha_n I+K^*K)^{-1}K^*W_i,\mu_1\right\rangle \\
&\xrightarrow{d} N(0,1).
\end{aligned}
$$

and the result follows provided that all other terms are asymptotically negligible. To see that the Lindeberg's condition in equation (A.2) is satisfied, note that for every $\delta > 0$,

$$\mathbb{E}\left[|\langle \eta_n, \mu_1\rangle|^2 \mathbb{1}_{\{\pi_n|\langle \eta_n,\mu_1\rangle|\geq \epsilon n\}}\right] \leq \frac{\pi_n^\delta}{\epsilon^\delta n^\delta}\mathbb{E}\,|\langle \eta_n, \mu_1\rangle|^{2+\delta}$$

and that $\pi_n \sim n^{-c}$ with $c \in (0, 1/2]$ depending on the mapping properties of operators K and Σ. Therefore, the Lindeberg condition is satisfied provided that $\mathbb{E}\,|\langle \eta_n, \mu_1\rangle|^{2+\delta} = O(1)$. This is easily verified under Assumption 4.1 since

$$\begin{aligned}|\langle \eta_n, \mu_1\rangle| &\lesssim |U + \langle Z, \varphi_0\rangle| \left\|(K^*K)^{\tilde{\gamma}}(\alpha_n I + K^*K)^{-1}K^*(K^*K)^\gamma\right\| \\ &\lesssim |U| + |\langle Z, \varphi_0\rangle|.\end{aligned}$$

Therefore, it remains to show that all other terms normalized with π_n are asymptotically negligible. For II_n', by the Cauchy-Schwartz inequality

$$\begin{aligned}II_n' &= \left\langle \frac{1}{n}\sum_{i=1}^n W_i(U_i + \langle Z_i, \varphi_0\rangle), \hat{K}^*\left((\alpha_n I + \hat{K}^*\hat{K})^{-1} - (\alpha_n I + K^*K)^{-1}\right)\mu_1\right\rangle \\ &\leq \left\|\frac{1}{n}\sum_{i=1}^n W_i(U_i + \langle Z_i, \varphi_0\rangle)\right\| \left\|\hat{K}^*(\alpha_n I + \hat{K}^*\hat{K})^{-1}(\hat{K}^*\hat{K} - K^*K)(\alpha_n I + K^*K)^{-1}\mu_1\right\|.\end{aligned}$$

Since $\mu_1 \in \mathcal{R}\left[(K^*K)^\gamma\right]$, there exists some $\psi \in \mathcal{E}$ such that $\mu_1 = (K^*K)^\gamma\psi$ and so

$$\begin{aligned}II_n' &\lesssim_P n^{-1/2}\left\|\hat{K}^*(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\| \left\|\hat{K} - K\right\| \left\|(\alpha_n I + K^*K)^{-1}(K^*K)^\gamma\psi\right\| \\ &\quad + n^{-1/2}\left\|\hat{K}^*(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\| \left\|\hat{K}^* - K^*\right\| \left\|K(\alpha_n I + K^*K)^{-1}(K^*K)^\gamma\psi\right\| \\ &\lesssim_P n^{-1}\left\|(\alpha_n I + K^*K)^{-1}(K^*K)^\gamma\psi\right\| + n^{-1}\alpha_n^{-1/2}\left\|K(\alpha_n I + K^*K)^{-1}(K^*K)^\gamma\psi\right\| \\ &\lesssim_P n^{-1}\alpha_n^{\gamma\wedge 1-1} + n^{-1}\alpha_n^{\gamma\wedge 1/2-1} = o_P(\pi_n^{-1}),\end{aligned}$$

where the last line follows under Assumption 4.2. Similarly,

$$\begin{aligned}III_n' &\leq \left\|\hat{K}^* - K^*\right\| \left\|\frac{1}{n}\sum_{i=1}^n W_i(U_i + \langle Z_i, \varphi_0\rangle)\right\| \left\|(\alpha_n I + K^*K)^{-1}(K^*K)^\gamma\psi\right\| \\ &\lesssim_P n^{-1}\alpha_n^{\gamma\wedge 1-1} \\ &= o_P(\pi_n^{-1}).\end{aligned}$$

Next, decompose

$$\begin{aligned}
IV_n' &= \left\langle (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\hat{K}\varphi_1 - (\alpha_n I + K^*K)^{-1}K^*K\varphi_1, \mu_1 \right\rangle \\
&= \left\langle \alpha_n(\alpha_n I + \hat{K}^*\hat{K})^{-1}\left[\hat{K}^*\hat{K} - K^*K\right](\alpha_n I + K^*K)^{-1}\varphi_1, \mu_1 \right\rangle \\
&= \left\langle (\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*(\hat{K} - K)b_n, \mu_1 \right\rangle \\
&\quad + \left\langle (\alpha_n I + \hat{K}^*\hat{K})^{-1}(\hat{K}^* - K^*)Kb_n, \mu_1 \right\rangle \\
&\triangleq IV_n^a + IV_n^b + IV_n^c + IV_n^d + IV_n^e
\end{aligned}$$

with

$$\begin{aligned}
IV_n^a &= \left\langle \left\{(\alpha_n I + \hat{K}^*\hat{K})^{-1} - (\alpha_n I + K^*K)^{-1}\right\}\hat{K}^*(\hat{K} - K)b_n, \mu_1 \right\rangle \\
IV_n^b &= \left\langle \left\{(\alpha_n I + \hat{K}^*\hat{K})^{-1} - (\alpha_n I + K^*K)^{-1}\right\}(\hat{K}^* - K^*)Kb_n, \mu_1 \right\rangle \\
IV_n^c &= \left\langle (\alpha_n I + K^*K)^{-1}K^*(\hat{K} - K)b_n, \mu_1 \right\rangle \\
IV_n^d &= \left\langle (\alpha_n I + K^*K)^{-1}(\hat{K}^* - K^*)(\hat{K} - K)b_n, \mu_1 \right\rangle \\
IV_n^e &= \left\langle (\alpha_n I + K^*K)^{-1}(\hat{K}^* - K^*)Kb_n, \mu_1 \right\rangle.
\end{aligned}$$

We bound the last three terms by the Cauchy-Schwartz inequality

$$\begin{aligned}
IV_n^c &\leq \left\|\hat{K} - K\right\| \|b_n\| \left\|K(\alpha_n I + K^*K)^{-1}\mu_1\right\| \lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge 1 + \gamma\wedge\frac{1}{2}}}{\sqrt{n\alpha_n}} \\
IV_n^d &\leq \left\|\hat{K}^* - K^*\right\| \left\|\hat{K} - K\right\| \|b_n\| \left\|(\alpha_n I + K^*K)^{-1}\mu_1\right\| \lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge 1 + \gamma\wedge 1}}{n\alpha_n} \\
IV_n^e &\leq \left\|\hat{K}^* - K^*\right\| \|Kb_n\| \left\|(\alpha_n I + K^*K)^{-1}\mu_1\right\| \lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge\frac{1}{2} + \gamma\wedge 1}}{\sqrt{n\alpha_n}}.
\end{aligned}$$

Next, for the first two terms, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
IV_n^a &\leq \left\|\hat{K} - K\right\|^2 \left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\| \|b_n\| \left\|K(\alpha_n I + K^*K)^{-1}\mu_1\right\| \\
&\quad + \left\|\hat{K}^* - K^*\right\| \left\|\hat{K}(\alpha_n I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\right\| \left\|\hat{K} - K\right\| \|b_n\| \left\|(\alpha_n I + K^*K)^{-1}\mu_1\right\| \\
&\lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge 1 + \gamma\wedge\frac{1}{2}}}{n\alpha_n} + \frac{\alpha_n^{\frac{\beta}{2}\wedge 1 + \gamma\wedge 1}}{n\alpha_n} \lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge 1 + \gamma\wedge\frac{1}{2}}}{n\alpha_n}
\end{aligned}$$

and

$$
\begin{aligned}
IV_n^b &\leq \left\|\hat{K} - K\right\| \left\|(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\| \left\|\hat{K}^* - K^*\right\| \|Kb_n\| \left\|K(\alpha_n I + K^*K)^{-1}\mu_1\right\| \\
&\quad + \left\|\hat{K}^* - K^*\right\|^2 \left\|\hat{K}(\alpha_n I + \hat{K}^*\hat{K})^{-1}\right\| \|Kb_n\| \left\|(\alpha_n I + K^*K)^{-1}\mu_1\right\| \\
&\lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge\frac{1}{2}+\gamma\wedge\frac{1}{2}}}{n\alpha_n} + \frac{\alpha_n^{\frac{\beta}{2}\wedge\frac{1}{2}+\gamma\wedge 1}}{n\alpha_n} \lesssim_P \frac{\alpha_n^{\frac{\beta}{2}\wedge\frac{1}{2}+\gamma\wedge\frac{1}{2}}}{n\alpha_n}.
\end{aligned}
$$

Lastly,

$$
\pi_n\langle b_n, \mu_1\rangle \lesssim \pi_n\|(K^*K)^\gamma b_n\| \lesssim \pi_n\alpha_n^{(\gamma+\beta/2)\wedge 1}.
$$

This completes the proof. □

A.3 Partial Identification

In this section, we discuss how our results can be used for partial identification; see Freyberger (2017) and Santos (2012). It is worth emphasizing that this section serves illustrative purposes only, and the comprehensive study of the partial identification approach is left for future research. Let

$$
|I_0| \triangleq \sup_{\phi_1,\phi_2\in I_0} \|\phi_1 - \phi_2\|_\infty
$$

be the L_∞-diameter of the identified set. Suppose that φ and φ_1 belong to the Lipschitz smoothness class

$$
\mathscr{F} \triangleq \left\{\phi \in L_\infty([0,1]^p):\ \|\phi\|_s \triangleq \|\phi\|_\infty + \sup_{z_1\neq z_2} \frac{|\phi(z_1) - \phi(z_2)|}{\|z_1 - z_2\|} \leq C\right\}.
$$

If the best approximation φ_1 and the diameter of the identified set $|I_0|$ are known, then $\varphi_1 \pm |I_0|$ is a valid identified set for the structural function φ. The results of the paper show that we can estimate the best approximation consistently. It remains to show that we can infer the diameter of the identified set.

To that end, for a fixed $\varepsilon > 0$, we focus on the following hypotheses

$$
H_0:\ |I_0| \geq \varepsilon \qquad \text{vs.} \qquad H_1:\ |I_0| < \varepsilon
$$

and follow Freyberger (2017) in the construction of test statistics. Note that under H_0, we have $\|\phi_1 - \phi_2\|_\infty \geq \varepsilon$ for some $\phi_1, \phi_2 \in I_0$. Let $\phi = \phi_1 - \phi_2$. Then $K\phi = 0$, $\|\phi\|_s \leq 2C$, and $\|\phi\|_\infty \geq \varepsilon$. This motivates the following statistics

$$
T_\varepsilon \triangleq \inf_{\phi:\|\phi\|_s\leq 2C, \|\phi\|_\infty\geq\varepsilon} \|K\phi\|^2 = \inf_{\phi\in\mathscr{F}_\varepsilon, \|\phi\|_\infty = 1} \|K\phi\|^2.
$$

where $\mathscr{F}_\varepsilon = \{\phi \in L_\infty([0,1]^p) : \|\phi\|_s \leq 2C/\varepsilon\}$. Clearly, under H_0, we have $T_\varepsilon = 0$. Let

$$\hat{T}_\varepsilon = \inf_{\phi \in \mathscr{F}_\varepsilon : \|\phi\|_\infty = 1} \|\hat{K}\phi\|^2$$

be the sample counterpart, where $\hat{K}$ is a consistent estimator of K.

For the NPIV regression, suppose that the density $\hat{f}_{ZW}$ is fitted using the series estimator, and the statistics $\hat{T}_\varepsilon$ is computed over the J-dimensional sieve. Freyberger (2017), Theorem 2, shows that

$$\hat{\varepsilon} = \sup\left\{\varepsilon \in [0, \bar{C}] : n\hat{T}_\varepsilon \leq q_{1-\alpha}\right\}$$

estimates consistently the upper bound on $|I_0|$, where $\bar{C}$ is the largest ε such that $\{\phi : \|\phi\|_s \leq 2C, \|\phi\|_\infty \geq \varepsilon\} \neq \emptyset$ and $q_{1-\alpha}$ is a certain critical value.

Therefore, combining this result with our result on the L_∞ consistency for the best approximation, the tube $[\hat{\varphi} - \hat{\varepsilon}, \hat{\varphi} + \hat{\varepsilon}]$ is a valid set estimator of φ.[11]

A.4 Extreme Nonidentification

In this section, we obtain an approximation of the large sample distribution of the Tikhonov-regularized estimators in extremely nonidentified cases. Interestingly, we show that the asymptotic distribution is a weighted sum of independent chi-squared random variables. This result will serve as a starting point for Section 4, where we document a certain transition between the chi-squared and the Gaussian limits in the intermediate cases.[12]

A.4.1 High-dimensional Regressions

In high-dimensional regressions, the identification strength is described by the covariance operator of Z and W. In the extremely nonidentified case, the covariance operator is degenerate, and we obtain the following result.

Theorem A.1. *Suppose that Assumption 4.1 is satisfied,* $\mathbb{E}[\langle Z, \delta\rangle W] = 0,\ \forall \delta \in \mathcal{E}$, *and* $\alpha_n n \to \infty$. *Then*

$$\alpha_n n(\hat{\varphi} - \varphi_1) \xrightarrow{d} \mathbb{E}\left[\|W\|^2 YZ\right] + J(h),$$

where $h(X, X') = \frac{1}{2}\langle W, W'\rangle(ZY' + Z'Y)$, $X' = (Y', Z', W')$ *is an independent copy of* $X = (Y, Z, W)$, *and J is a stochastic Wiener-Itô integral.*

[11] We are grateful to the anonymous referee for pointing out this connection.

[12] Extreme nonidentification also relates to the weak instruments problem.

Note that the theorem states weak convergence in the topology of the Hilbert space $\mathcal{E}$, which is impossible to achieve in the regularly identified case. It can be shown that the distribution of inner products of $J(h)$ with $\mu \in \mathcal{E}$ is a weighted sum of chi-squared random variables. Also, interestingly, Theorem A.1 does not require that $\alpha_n \to 0$ as $n \to \infty$.

Proof of Theorem A.1. Since $\mathbb{E}[\langle Z, \delta \rangle W] = 0$ for all $\delta \in \mathcal{E}$, we have $\varphi_1 = 0$. Then

$$\alpha_n n \left(\hat{\varphi} - \varphi_1\right) = \left(I + \frac{1}{\alpha_n} \hat{K}^* \hat{K}\right)^{-1} n \hat{K}^* \hat{r}.$$

Under Assumption 4.1

$$\mathbb{E}\|\hat{K}\|^2 = \mathbb{E}\|\hat{K} - K\|^2 \leq \mathbb{E}\left\| \frac{1}{n} \sum_{i=1}^n Z_i W_i - \mathbb{E}[ZW] \right\|^2 = O\left(\frac{1}{n}\right).$$

Then $\|\hat{K}^* \hat{K}\| \leq \|\hat{K}\|^2 = O_P\left(n^{-1}\right)$. Therefore, as $\alpha_n n \to \infty$, by the continuous mapping theorem; see van der Vaart and Wellner (2000), Theorem 1.3.6

$$\alpha_n n \left(\hat{\varphi} - \varphi_1\right) = \left(I + o_P(1)\right)^{-1} n \hat{K}^* \hat{r}.$$

By Slutsky's theorem, see van der Vaart and Wellner (2000), Example 1.4.7, it suffices to obtain the asymptotic distribution of $n\hat{K}^* \hat{r}$.

Note that

$$\begin{aligned} n\hat{K}^* \hat{r} &= \frac{1}{n} \sum_{i,j=1}^n \langle W_i, W_j \rangle Z_i Y_j \\ &= \frac{1}{n} \sum_{i=1}^n \|W_i\|^2 Z_i Y_i + \frac{1}{n} \sum_{i \neq j} \langle W_i, W_j \rangle Z_i Y_j. \end{aligned}$$

Under Assumption 4.1, by the Mourier law of large numbers

$$\frac{1}{n} \sum_{i=1}^n \|W_i\|^2 Z_i Y_i \xrightarrow{a.s.} \mathbb{E}\left[\|W\|^2 ZY\right].$$

Since $\mathbb{E}[\langle Z, \delta \rangle W] = 0, \forall \delta \in \mathcal{E}$, the second term is a Hilbert space-valued degenerate U-statistics

$$\begin{aligned} n\mathbf{U}_n &\triangleq \frac{1}{n} \sum_{i \neq j} \langle W_i, W_j \rangle Z_i Y_j \\ &= \frac{2}{n} \sum_{i<j} \frac{Z_i Y_j + Z_j Y_i}{2} \langle W_i, W_j \rangle. \end{aligned}$$

Under the Assumption 4.1, by the Borovskich CLT, see Theorem B.1

$$n\mathbf{U}_n \xrightarrow{d} J(h),$$

where $J(h) = \iint_{\mathcal{X}\times\mathcal{X}} h(x_1, x_2)\mathbb{W}(\mathrm{d}x_1)\mathbb{W}(\mathrm{d}x_2)$ is a stochastic Wiener-Itô integral, $\mathbb{W}$ is a Gaussian random measure on $\mathcal{X}$, $h(X, X') = \frac{ZY'+Z'Y}{2}\langle W, W'\rangle$, and $X' = (Y', Z', W')$ is an independent copy of $X = (Y, Z, W)$. □

A.4.2 Nonparametric IV Regression

In nonparametric IV regression, the identification strength is described by the conditional expectation operator. In the extreme nonidentified case,

$$\mathbb{E}[\phi(Z)|W] = 0, \qquad \forall \phi \in L_{2,0}(Z),$$

where $L_{2,0}(Z) = \{\phi \in L_2(Z) : \mathbb{E}\phi(Z) = 0\}$, so that K is a degenerate conditional expectation operator. Consider the operator $T : \phi \mapsto \mathbb{E}_X[\phi(X)h(X, X')]$ on $L_2(X)$, where $\mathbb{E}_X$ is expectation with respect to $X = (Y, Z, W)$ only, X' is an independent copy of X, and

$$h(x, x') = \frac{1}{2}\left\{yP_0\mu(z') + y'P_0\mu(z)\right\} h_w^{-q}\bar{K}\left(h_w^{-1}(w - w')\right).$$

Here and later, P_0 is the projection operator on $L_{2,0}$ and $\bar{K}(v) = \int K_w(v-u)K_w(u)\mathrm{d}u$ is the convolution kernel. The following assumption is a set of mild restrictions on the distribution of the data:

Assumption A.4.1. *(i) $(Y_i, Z_i, W_i)_{i=1}^n$ is an i.i.d. sample of (Y, Z, W); (ii) $\mathbb{E}[|Y||Z] < \infty$, $\mathbb{E}[|Y|^2|W] < \infty$ a.s.; (iii) $K_j \in L_1 \cap L_2, j \in \{z, w\}$ and K_w is a symmetric and bounded function; (iv) $f_Z \in L_\infty$.*

Let h_z and h_w be the bandwidth parameters smoothing respectively over Z and W. The following result holds:

Theorem A.2. *Suppose that Assumption A.4.1 is satisfied, $\mathbb{E}[\phi(Z)|W] = 0, \forall \phi \in L_{2,0}(Z)$, and $n\alpha_n h_z^p \to \infty$ with h_w being fixed. Then for every $\mu \in L_2([0, 1]^p)$*

$$\alpha_n n\langle\hat{\varphi} - \varphi_1, \mu\rangle \xrightarrow{d} \mathbb{E}\left[YP_0\mu(Z)\right] h_w^{-q}\bar{K}(0) + \sum_{j=1}^{\infty}\lambda_j(\chi_j^2 - 1),$$

where $(\chi_j^2)_{j\geq 1}$ are independent chi-squared random variables with 1 degree of freedom and $(\lambda_j)_{j\geq 1}$ are eigenvalues of T.

Proof of Theorem A.2. Since $\mathbb{E}[\phi(Z)|W] = 0, \forall \phi \in L_{2,0}(Z)$, we have $\varphi_1 = 0$. Note also that the adjoint operator to K is $P_0 K^*$, where P_0 is the orthogonal projection on $L_{2,0}(Z)$. Then

$$\alpha_n n(\hat{\varphi} - \varphi_1) = \left(I + \frac{1}{\alpha_n} P_0 \hat{K}^* \hat{K}\right)^{-1} n P_0 \hat{K}^* \hat{r},$$

where $\hat{P}_0$ is the estimator of P_0. Under Assumption A.4.1 (i) since $\mathbb{E}[\phi(Z)|W] = 0$ for all $\phi \in L_{2,0}(Z)$

$$\begin{aligned}
\mathbb{E}\left\|P_0 \hat{K}^* \hat{K}\right\| &\leq \mathbb{E}\|P_0 \hat{K}\|^2 \leq \mathbb{E}\|P_0 \hat{f}_{ZW}\|^2 \\
&= \mathbb{E}\left\|\frac{1}{n h_z^p h_w^q} \sum_{i=1}^n P_0 K_z \left(h_z^{-1}(Z_i - z)\right) K_w \left(h_z^{-1}(W_i - w)\right)\right\|^2 \\
&\leq \frac{1}{n h_z^{2p} h_w^{2q}} \mathbb{E}\left\|P_0 K_z \left(h_z^{-1}(Z_i - z)\right) K_w \left(h_z^{-1}(W_i - w)\right)\right\|^2 \\
&= \frac{1}{n h_z^p h_w^q} \|P_0 K_z\| \|K_w\| = O\left(\frac{1}{n h_z^p}\right).
\end{aligned}$$

Therefore, $\frac{1}{\alpha_n}\left\|\hat{P}_0 \hat{K}^* \hat{K}\right\| = o_P(1)$ as $n\alpha_n h_z^p \to \infty$. Then by the continuous mapping and the Slutsky's theorems, it suffices to characterize the asymptotic distribution of

$$n P_0 \hat{K}^* \hat{r} = \frac{1}{n h_z^p h_w^q} \sum_{i,j} Y_i P_0 K_z \left(h_z^{-1}(Z_j - z)\right) \bar{K}\left(h_w^{-1}(W_i - W_j)\right).$$

To that end, for every $\mu \in L_2([0,1]^p)$

$$\begin{aligned}
\left\langle n P_0 \hat{K}^* \hat{r}, \mu\right\rangle &= \left\langle n \hat{K}^* \hat{r}, P_0 \mu\right\rangle \\
&\triangleq \zeta_n + \mathbf{U}_n + R_n
\end{aligned}$$

with

$$\begin{aligned}
\zeta_n &= \frac{1}{n} \sum_{i=1}^n Y_i P_0 \mu(Z_i) h_w^{-q} \bar{K}(0), \\
\mathbf{U}_n &= \frac{2}{n} \sum_{i<j} \frac{1}{2} \left\{Y_i P_0 \mu(Z_j) + Y_j P_0 \mu(Z_i)\right\} h_w^{-q} \bar{K}\left(h_w^{-1}(W_i - W_j)\right), \\
R_n &= \frac{1}{n h_w^q} \sum_{i,j=1}^n Y_i \left\{[K_z * P_0 \mu](Z_j) - P_0 \mu(Z_j)\right\} \bar{K}\left(h_n^{-1}(W_i - W_j)\right),
\end{aligned}$$

where $[K_z * P_0\mu](z) = h_n^{-p} \int K\left(h_z^{-1}(z-u)\right) P_0\mu(u)\mathrm{d}u$. Under Assumption A.4.1, by the strong law of large numbers

$$\zeta_n \xrightarrow{a.s} \mathbb{E}\left[Y P_0\mu(Z)\right] h_w^{-q} \bar{K}(0).$$

Since $\mathbb{E}[\phi(Z)|W] = 0, \forall \phi \in L_{2,0}(Z)$, $\mathbf{U}_n$ is a centered degenerate U-statistics. By the central limit theorem for the degenerate U-statistics, see Gregory (1977),

$$\mathbf{U}_n = \frac{2}{n}\sum_{i<j} h(X_i, X_j) \xrightarrow{d} \sum_{j=1}^{\infty} \lambda_j(\chi_j^2 - 1).$$

Lastly, decompose $R_n = R_{1n} + R_{2n}$ with

$$\begin{aligned}
R_{1n} &= \frac{1}{n}\sum_{i=1}^{n} Y_i \left\{[K_z * P_0\mu](Z_i) - P_0\mu(Z_i)\right\} h_w^{-q} \bar{K}(0), \\
R_{2n} &= \frac{1}{n}\sum_{i<j} Y_i \left\{[K_z * P_0\mu](Z_j) - P_0\mu(Z_j)\right\} h_w^{-q} \bar{K}\left(h_w^{-1}(W_i - W_j)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{E}|R_{1n}| &\leq \mathbb{E}\left|Y\left\{[K_z * P_0\mu](Z) - P_0\mu(Z)\right\}\right| h_w^{-q}\bar{K}(0) \\
&\lesssim \int \left|[K_z * P_0\mu](z) - P_0\mu(z)\right| f_Z(z)\mathrm{d}z \\
&\leq \|K_z * \mu - \mu\|^2 \|f_Z\|^2 \\
&= o(1),
\end{aligned}$$

where the first two lines follow under Assumption A.4.1 (i)-(ii), the third by the Cauchy-Schwartz inequality and $\|P_0\| \leq 1$, and the last by Giné and Nickl (2016), Proposition 4.1.1. (iii). Similarly, since $\mathbb{E}\left[|Y|^2|W\right] < \infty$ a.s. and $\bar{K} \in L_\infty$, by the moment inequality in Korolyuk and Borovskich (1994), Theorem 2.1.3

$$\begin{aligned}
\mathbb{E}|R_{2n}|^2 &\lesssim \mathbb{E}\left|Y\left\{[K_z * P_0\mu](Z') - P_0\mu(Z')\right\} h_w^{-q}\bar{K}\left(h_w^{-1}(W - W')\right)\right|^2 \\
&\lesssim \int \left|[K_z * P_0\mu](z) - P_0\mu(z)\right| f_Z(z)\mathrm{d}z = o(1).
\end{aligned}$$

$\square$

ONLINE APPENDIX

B.1 Generalized Inverse

In this section, we collect some facts about the generalized inverse operator from operator theory; see also Carrasco, Florens, and Renault (2007) for a comprehensive review of different aspects of the theory of ill-posed inverse models in econometrics. Let $\varphi \in \mathcal{E}$ be a structural parameter in a Hilbert space $\mathcal{E}$ and let $K : \mathcal{E} \to \mathcal{H}$ be a bounded linear operator mapping to a Hilbert spaces $\mathcal{H}$. Consider the functional equation

$$K\varphi = r.$$

If the operator K is not one-to-one, then the structural parameter φ is not point identified, and the identified set is a closed linear manifold described as $\Phi^{\mathrm{ID}} = \varphi + \mathcal{N}(K)$, where $\mathcal{N}(K) = \{\phi : K\phi = 0\}$ is the null space of K; see Figure B.1. The following result offers equivalent characterizations of the identified set; see Groetsch (1977), Theorem 3.1.1 for a formal proof.

Proposition B.1.1. *The identified set I_0 is characterized as a set of solutions to*

(i) the least-squares problem: $\min_{\phi\in\mathcal{E}} \|K\phi - r\|$;

(ii) the normal equations: $K^*K\phi = K^*r$, *where* K^* *is the adjoint operator to* K.

The generalized inverse is formally defined below.

Definition B.1.1. *The generalized inverse of the operator K is a unique linear operator $K^\dagger : \mathcal{R}(K) \oplus \mathcal{R}(K)^\perp \to \mathcal{E}$ defined by $K^\dagger r = \varphi_1$, where $\varphi_1 \in I_0$ is a unique solution to*

$$\min_{\phi\in I_0} \|\phi\|. \tag{A.1}$$

For nonidentified linear models, the generalized inverse maps r to the unique minimal norm element of I_0. It follows from equation (A.1) that φ_1 is a projection of 0 on the identified set. Therefore, φ_1 is the projection of the structural parameter φ on the orthogonal complement to the null space $\mathcal{N}(K)^\perp$, see Figure B.1, and we call φ_1 the best approximation to the structural parameter φ. The generalized inverse operator is typically a discontinuous map, as illustrated in the following proposition; see Groetsch (1977), pp.117-118 for more details.

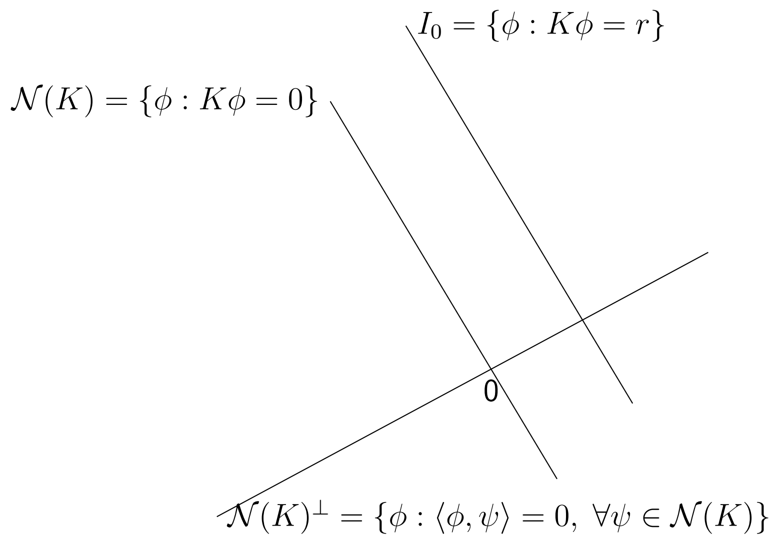

Figure B.1: Fundamental subspaces of $\mathcal{E}$.

Proposition B.1.2. *Suppose that the operator K is compact. Then the generalized inverse $K^\dagger$ is continuous if and only if $\mathcal{R}(K)$ is finite-dimensional.*

The following example illustrates this when K is an integral operator on spaces of square-integrable functions.

Example B.1.1. *Suppose that K is an integral operator*

$$
\begin{aligned}
K : L_2 &\to L_2 \\
\phi &\mapsto \int \phi(z)k(z,w)\mathrm{d}z.
\end{aligned}
$$

Then K is compact whenever the kernel function k is square integrable. In this case, the generalized inverse is continuous if and only if k is a degenerate kernel function

$$k(z,w) = \sum_{j=1}^{m} \phi_j(z)\psi_j(w).$$

It is worth stressing that in the NPIV model, the kernel function k is typically a non-degenerate probability density function. Moreover, in econometric applications, r is usually estimated from the data, so that $K^\dagger \hat{r} \xrightarrow{p} K^\dagger r = \varphi_1$ may not hold even when $\hat{r} \xrightarrow{p} r$ due to the discontinuity of $K^\dagger$.[13] In other words, we are faced with an ill-posed inverse problem. Tikhonov regularization can be understood as a method that smooths out the discontinuities of the generalized inverse $(K^*K)^\dagger$.[14]

B.2 Degenerate U-statistics in Hilbert Spaces

B.2.1 Wiener-Itô Integral

In this section, we review results on the asymptotic distribution of degenerate U-statistics in Hilbert spaces. Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space, and let H be a separable Hilbert space. We use $L_2(\mathcal{X}^m, H)$ to denote the space of all functions $f : \mathcal{X}^m \to H$ such that $\mathbb{E}\|f(X_1, \ldots, X_m)\|^2 < \infty$. The stochastic process $\{\mathbb{W}(A), A \in \Sigma_\mu\}$ indexed by the sigma-field $\Sigma_\mu = \{A \in \Sigma : \ \mu(A) < \infty\}$ is called the *Gaussian random measure* if

[13] In practice, the situation is even more complex because the operator K is also estimated from the data.

[14] By Proposition B.1.1, solving $K\varphi = r$ is equivalent to solving $K^*K\varphi = K^*r$. The latter is more attractive to work with because the spectral theory of self-adjoint operators in Hilbert spaces applies to K^*K.

1. For all $A \in \Sigma_\mu$
$$\mathbb{W}(A) \sim N(0, \mu(A)).$$

2. For any collection of disjoint sets $(A_k)_{k=1}^K$ in Σ_μ, $\mathbb{W}(A_k), k = 1, \dots, K$ are independent, and
$$\mathbb{W}\left(\bigcup_{k=1}^K A_k\right) = \sum_{k=1}^K \mathbb{W}(A_k).$$

Let $(A_k)_{k=1}^K$ be pairwise disjoint sets in Σ_μ, and let S_m be a set of simple functions $f \in L_2(\mathcal{X}^m, H)$ such that

$$f(x_1, \dots, x_m) = \sum_{i_1,\dots,i_m=1}^K c_{i_1,\dots,i_m} \mathbb{1}_{A_{i_1}}(x_1) \times \dots \times \mathbb{1}_{A_{i_m}}(x_m),$$

where $c_{i_1,\dots,i_m}$ is zero if any two indices $i_1, \dots, i_m$ are equal, i.e., f vanishes on the diagonal. For a Gaussian random measure $\mathbb{W}$ corresponding to P, consider the following random operator $J_m : S_m \to H$

$$J_m(f) = \sum_{i_1,\dots,i_m=1}^K c_{i_1,\dots,i_m} \mathbb{W}(A_{i_1}) \dots \mathbb{W}(A_{i_m}).$$

The following three properties are immediate from the definition of J_m:

1. Linearity;

2. $\mathbb{E} J_m(f) = 0$;

3. Isometry: $\mathbb{E}\langle J_m(f), J_m(g)\rangle_H = \langle f, g\rangle_{L_2(\mathcal{X}^m, H)}$.

The set S_m is dense in $L_2(\mathcal{X}^m, H)$ and J_m can be extended to a continuous linear isometry on $L_2(\mathcal{X}^m, H)$, called the Wiener-Itô integral.

Example B.2.1. *Let $(B_t)_{t\geq 0}$ be a real-valued Brownian motion. Then for any $(t, s] \subset [0, \infty)$, $\mathbb{W}((t, s]) = B_s - B_t$ is a Gaussian random measure (where μ is the Lebesgue measure) with the Wiener-Itô integral $J : L_2([0, \infty), \mathrm{d}t) \to \mathbf{R}$ defined as $J(f) = \int f(t) \mathrm{d}B_t$.*

B.2.2 Central Limit Theorem

Let $(\mathcal{X}, \Sigma, P)$ be a probability space, where $\mathcal{X}$ is a separable metric space, and Σ is a Borel σ-algebra. Let $(X_i)_{i=1}^n$ be i.i.d. random variables taking values in $(\mathcal{X}, \Sigma, P)$. Consider some symmetric function $h : \mathcal{X} \times \mathcal{X} \to H$, where H is a separable Hilbert space. The H-valued U-statistics of degree 2 is defined as

$$\mathbf{U}_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h(X_i, X_j).$$

The U-statistics is called degenerate if $\mathbb{E}h(x_1, X_2) = 0$. The following result provides the limiting distribution of the degenerate H-valued U-statistics; see Korolyuk and Borovskich (1994), Theorem 4.10.2 for a formal proof.

Theorem B.1. *Suppose that $\mathbf{U}_n$ is a degenerate U-statistics such that $\mathbb{E}h(X_1, X_2) = 0$ and $\mathbb{E}\|h(X_1, X_2)\|^2 < \infty$. Then*

$$n\mathbf{U}_n \xrightarrow{d} J(h),$$

where $J(h) = \iint_{\mathcal{X} \times \mathcal{X}} h(x_1, x_2) \mathbb{W}(\mathrm{d}x_1) \mathbb{W}(\mathrm{d}x_2)$ is a stochastic Wiener-Itô integral, and $\mathbb{W}$ is a Gaussian random measure on H.